Infinite Intersections of Doubling Measures, Weights, and Function Classes

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Consider a function $f : [0,1] \to [0,\infty)$.

Definition. We say f is **doubling** if there exists a doubling constant $C < \infty$ such that

$$\int_{I_1} f\,dx \le C \int_{I_2} f\,dx \qquad (1)$$

for all intervals I_1, I_2 that are adjacent and of equal length. *Motivation.* Doubling prevents f from changing too much on all scales.

Definition. We say f is **n-adic doubling** if I_1, I_2 are restricted to n-adic siblings.

Definition. Let $n \ge 2$. An **n-adic interval** I takes form

$$I = [\frac{k-1}{n^m}, \frac{k}{n^m}) \text{ for } m, k \in \mathbb{Z} \qquad (2)$$

Each n-adic interval (parent) can be partitioned into n consecutive n-adic intervals (children) of equal length. These n intervals are siblings relative to each other.

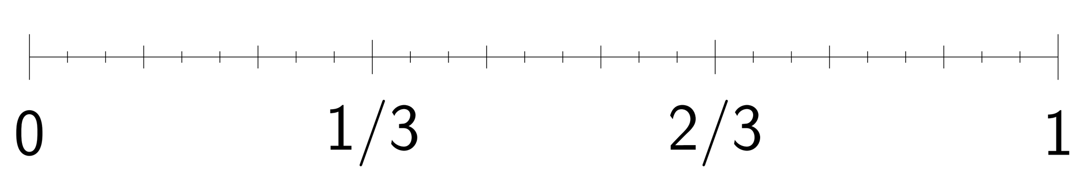

Figure: Triadic intervals

Question. Can f be n-adic doubling for all n, but not doubling? That is, intuitively can f be well-behaved on all n-adic intervals but overall poorly-behaved?

Lemma (ABMPZ)

Let $N \in \mathbb{N}$, $\epsilon > 0$. Then, there exists infinitely many $x \in \mathbb{N}$ such that

$$\left| \frac{1}{2^x} - \frac{1}{n_i^{[x \log_n 2]}} \right| < \frac{\epsilon}{2^x} \text{ for } 3 \le n \le N. \qquad (3)$$

Theorem (APRY)

Let $C_n > 1$ be unbounded. There exists infinitely many f that are n-adic doubling with n-adic constant at most C_n for each $n \ge 2$, but not doubling.

Claim. Start with $f \equiv 1$, perform a reweighting procedure for α steps with $\kappa > 1$. Then, f's doubling constant is at least κ^α. However, f is n-adic doubling with constant at most $1.01\kappa^3(2n)^{\log_2 \kappa}$ for $n \le N$.

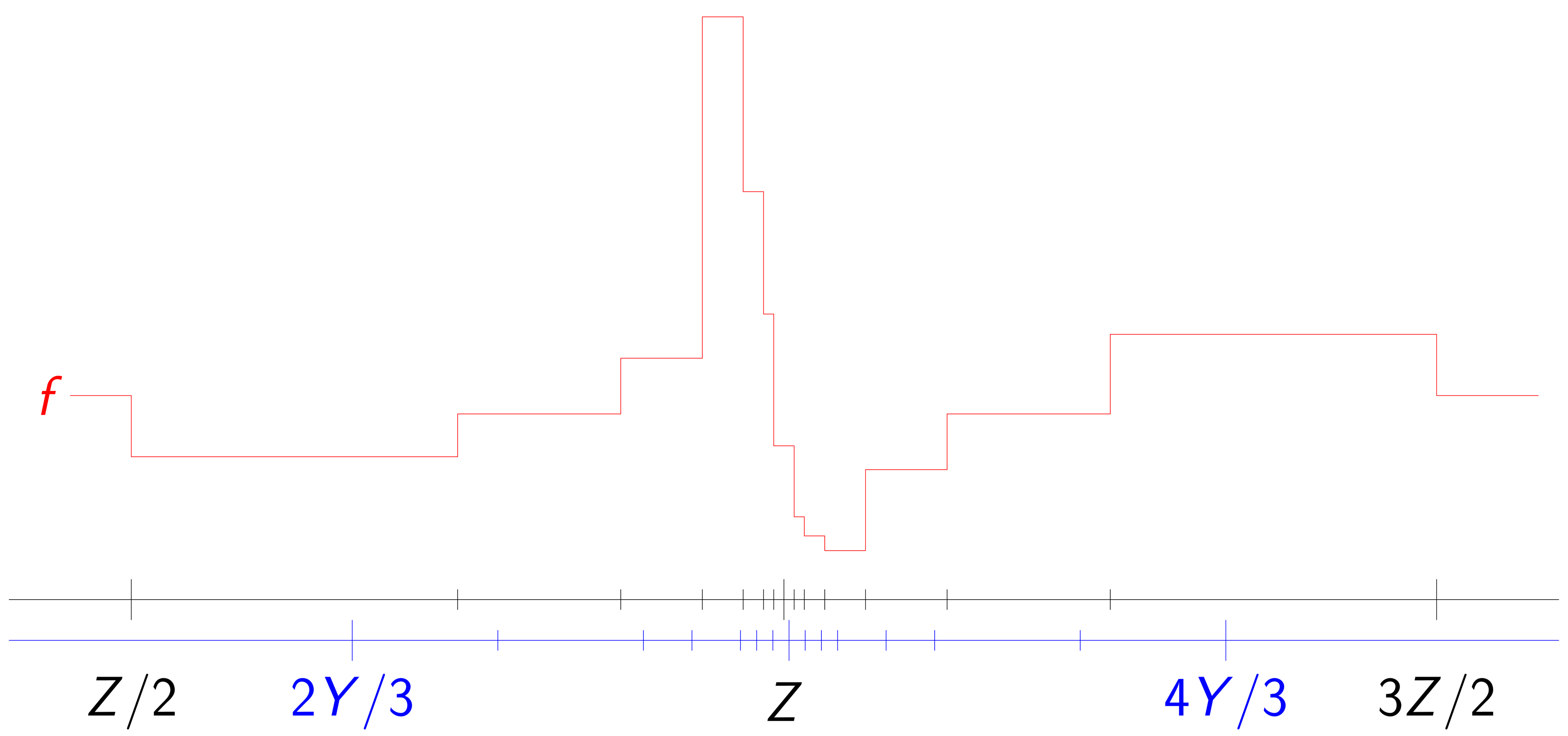

Figure: Three step reweighting procedure, $\kappa = 13/7$, $Z = \frac{1}{2^x}$, $Y = \frac{1}{3^{\lfloor x \log_3 2 \rfloor}}$.

Sketch. Reweighting preserves $\|f\|_{L^1}$. n-adic intervals larger than Y will not see reweighting. n-adic intervals smaller than Y will not cross Y and can only see at most $\sim \log_2(2n)$ many reweightings.

Claim. We perform infinitely many reweighting procedures while increasing N, α. Then, we can make $C = \infty$ while keeping $C_n \le 1.01\kappa^3(2n)^{\log_2 \kappa}$ for all $n \in \mathbb{N}$. If we also decrease κ, then we can make C_n any increasing, unbounded function.

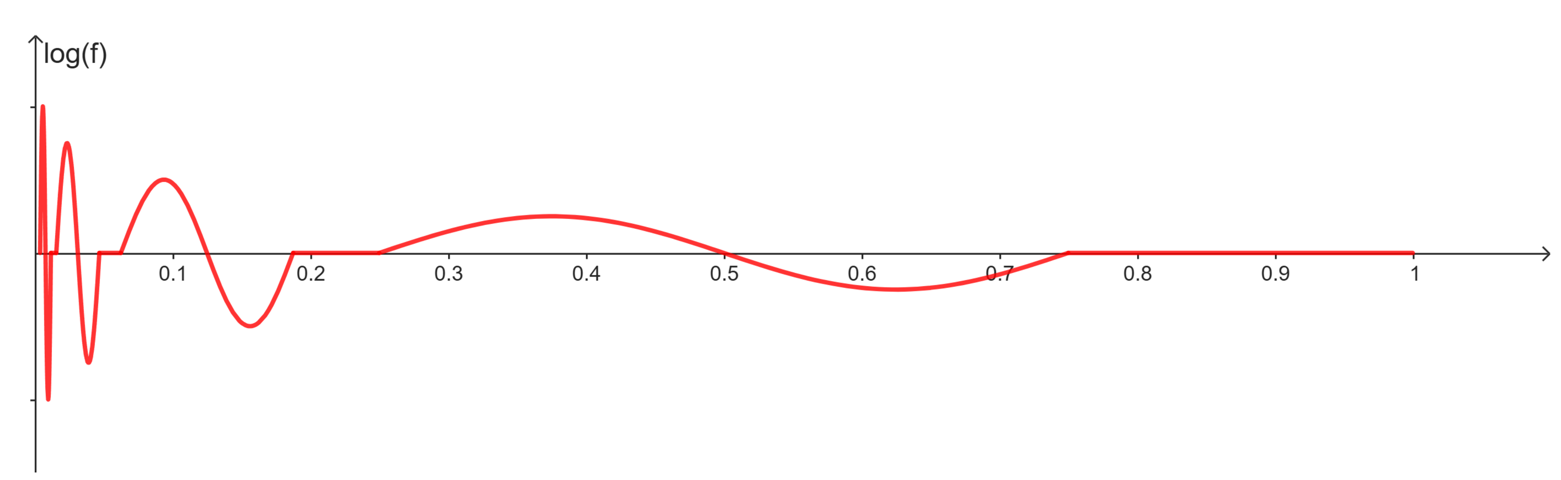

Figure: Infinitely many reweighting, qualitatively

Definition. We say that f satisfies a **reverse Holder** inequality, or $f \in RH^p$, if there exists $1 < p < \infty$, $C \ge 1$ such that

$$\int_I |f|^p\,dx \le C\left(\int_I |f|\,dx\right)^p \qquad (4)$$

for all intervals I. We say $f \in RH^p_n$ if we restrict I to n-adic intervals.

Definition. We say f is of **bounded mean oscillation**, or $f \in BMO$ if

$$\|f\|_{BMO} := \sup_I ⨍_I |f - ⨍_I f| < \infty \qquad (5)$$

Motivation. Like doubling, these conditions provide regularity for a function f.

It is known that if $f \in RH^p_n$ for some $1 < p < \infty$, then $\log|f| \in BMO_n$. But if f is not doubling, then $\log|f| \notin BMO$.

Claim. The previous f also satisfies a reverse Holder inequality for n-adic intervals.

Sketch. For intervals within a reweighting interval, we estimate a geometric series. For intervals crossing multiple reweighting, we need to glue the reverse Holder constants together.

Theorem (APRY)

The following holds: $\bigcap_{n\ge 2} RH^p_n \subsetneq RH^p$ for all $1 \le p < \infty$ and $\bigcap_{n \ge 2} BMO_n \subsetneq BMO$.

By duality we also have similar results for to the Hardy space H^1 and the space of vanishing mean oscillation VMO.

- T.C. Anderson, E. Bellah, Z. Markman, T. Pollard and J. Zeitlin. Arbitrary finite intersections of doubling measures and applications. To appear in *Journal of Functional Analysis.*
- D.M Boylan, S.J. Mills, and L.A. Ward. Construction of an exotic measure: dyadic doubling and triadic doubling does not imply doubling. *J. Math. Anal. Appl.* 476 (2019), no. 2, 241–277.