Ideas and Insight in Synthetic Geometry

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Contents

Preface

This handouts organizes notable thoughts into "ideas", which are more general then theorems. They may illustrate particular geometric configurations, constructions, or ways of thinking.

Note that solutions to examples are not contained in the main text portion, in hopes that you first attempt the examples by yourself. After which, it is also strongly recommended that you read through all solutions, since this is where application of ideas are shown. (Basically, what are usually in the main text of math books - working through examples - are moved to the solution pages)

To draw connections between geometric ideas, many examples and problems will depend on results from previous examples. Therefore, the handout should be worked on in order. It is okay to not have fully solved all problems: understanding the problem after seeing hints or the solution is also beneficial.

Finally, for this handout no advanced techniques will be used. It will not be too rigorous with proofs. Solutions are written so that the reader may understand the underlying geometry ideas.

This handout should be considered as a work in progress. Please contact me regarding errors, clarifications, or comments of this handout.

Have fun doing geometry!

1 Isosceles Triangles

The most basic property in geometry is congruence $\cong$, which births congruent segments and congruent angles. With only congruence and angle addition, we are able to attempt many problems broadly known under the category of angle chasing.

We begin by analyzing the simplest of such figures, the isosceles triangle, and end with figures containing multiple isosceles triangles, such as the equilateral triangle and the cyclic quadrilateral.

This chapter serves as an introduction for identifying congruent segments/angles and constructing supplemental figures involving congruent triangles to complete problems.

Triangle Congruence

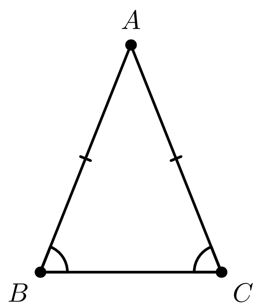


Figure 1: An isosceles triangle

Idea 1 (Pons asinorum). *Given 3 non-collinear points A, B, C, $AB = AC$ if and only if $\angle ABC = \angle ACB$.*

Of course, this statement seems to be obviously true by symmetry. However, we also need to make sure that the configuration is well-defined, which comes from the SAS triangle congruence condition. We know that $AB = AC$, $AC = AB$, and $\angle BAC = \angle CAB$, from which $\triangle BAC \cong \triangle CAB$ follows. Thus, $\angle ABC = \angle ACB$. Similarly, we may use AAS congruence for proving the opposite direction.

Remark 1.1. *In geometry, we have two notable definition/properties that we generally take as a postulate. The first one is that corresponding angles formed by a transversal cutting two parallel lines are congruent. The second is triangle congruence conditions.*

Recall the triangle congruence conditions, abbreviated as SSS, SAS, ASA, AAS, and HL, referring to congruent corresponding parts.

Triangle congruence alone is a powerful tool to prove existences.

Example 1.1. (Existence of Circumcenter and Incenter) Show that the three perpendicular bisectors of a triangle concur at one point. Similarly, show that the three angle bisectors of a triangle concur at one point.

Isosceles Triangles

With the basic definition of the isosceles triangle, along with angle measurements, we may solve many introductory geometry problems.

Example 1.2. (2007 AMC 12A) Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of BAD?

For the next two examples, we can't simply add and subtract from known angles. Instead, we will need to use algebra.

Idea 2 (Algebra)**.** *Problem solving is very much a backwards strategy. Setting up variables and equations is a powerful method to demand results satisfying certain properties.*

It is recommended, especially for this chapter, for you to draw large diagrams and label all known angles with measurements and add variables as necessary. This way, you can easily keep track of what you know, and also observe nice relations between angles.

Example 1.3. (1994 AHSME) In triangle ABC, $AB = AC$. If there is a point P strictly between A and B such that $AP = PC = CB$, then $\angle A =$?

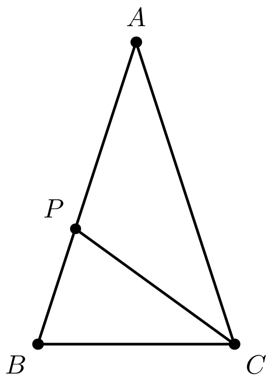


Figure 2: Example 1.3

Example 1.4. In triangle ABC, point D is on AC such that $AB = AD$. Suppose that $\angle ACB - \angle ABC = 30°$. Find $\angle CBD$.

By now, the only other facts we used are the sum of interior angles of a triangle is 180° and the angle addition postulate.

Remark 1.2. *At this point, tools introduced in the standard high school geometry class like vertical angles, exterior angles, and sum of interior angles in a triangle all boil down to the same idea that the "angle" of a straight line is 180 degrees. (Make sure you see why!) Thus, we wouldn't be explicitly referencing these results.*

However, many problems can not be solved so directly. Consider taking a peak at **Example 1.9** and **Example 1.11**. Try to label all known angles. You will find that there is a region in each diagram missing pairs of angles, for which we can only find the sum of the two measures.

While we are given familiar conditions, such as congruent segments in **Example 1.9** and congruent angles in **Example 1.11**, we don't see any isosceles triangles. The trick is that both problems have 3 hidden isosceles triangles, and it is our goal to be able to find them.

Angle Chasing

Idea 3 (Construction I)**.** *Constructing congruent triangles with different locations/orientations is a very powerful technique to manipulate congruent segments so that they form meaningful isosceles triangles.*

This is an overarching idea in this section. To construct new triangles, we can think forward with planar transformations like reflections and rotations. Alternatively, we can think backwards and draw triangles out of the congruent segments and then demand the triangles to be congruent.

We will begin illustrating this idea with very specific configurations.

Idea 4 (Utilize Symmetry)**.** *When we have highly symmetric figures like squares or equilateral triangles, a good idea is to rotate or reflect the figure and seeing what happens to newly connected components.*

In the next two examples, by rotating/reflecting the squares or triangles we can connect some segments/angles and also create $60°$ angles.

Idea 5 ($60°$ Angles)**.** *Be aware of $60°$ angles. They are the only special angle in angle chasing, and allow for more isosceles triangles to be formed.*

Example 1.5. In equilateral triangle ABC there is a point P such that $AP = 2BP$ and $\angle APB = 120°$. Find the measure of $\angle CPA$. **Hint**: 73

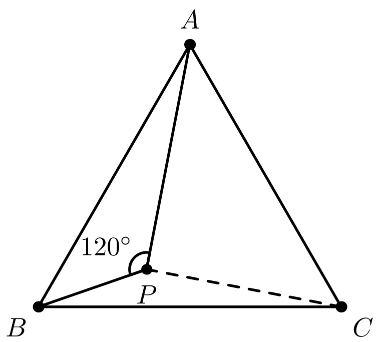


Figure 3: Example 1.5

Example 1.6. Given square $ABCD$ and point E inside the square such that $\angle EDC = \angle ECD = 15°$, prove that $\triangle AEB$ is equilateral. **Hint**: 68

Idea 6 (Reflection in Isosceles Triangles)**.** *In an isosceles triangle, if there are asymmetrical configurations, reflecting part of the configuration over the line of symmetry of the isosceles triangle can yield new observations.*

This is similar to the previous idea, but here we utilize the inherent symmetry of the isosceles triangle.

Example 1.7. In square $ABCD$, M is the midpoint of BC. The line through M perpendicular to AM meets CD at N. Show that $\angle BAM = \angle NAM$.

Example 1.8. (PUMaC 2013) Given triangle ABC and point P inside it, $\angle BAP = 18°$, $\angle CAP = 30°$, $\angle ACP = 48°$, and $AP = BC$. If $\angle BCP = x°$, find x. **Hint**: 20

Now, we will examine scenarios with multiple isosceles triangles, beginning with a beautiful example of multiple congruent segments.

Example 1.9. (2008 AMC 10B) Quadrilateral ABCD has $AB = BC = CD$, $\angle ABC = 70°$ and $\angle BCD = 170°$. What is the measure of angle BAD?

Cyclic Quadrilaterals

It turns out that in the last example, we actually have four equivalent segments, which the forth was created by a $60°$ angle. This new equal side can then come back and form more isosceles triangles with the previous two sides. This suggests that congruence $\cong$ is powerful because it is transitive.

Next, we will examine another configuration with four congruent segments, except that they all share a vertex.

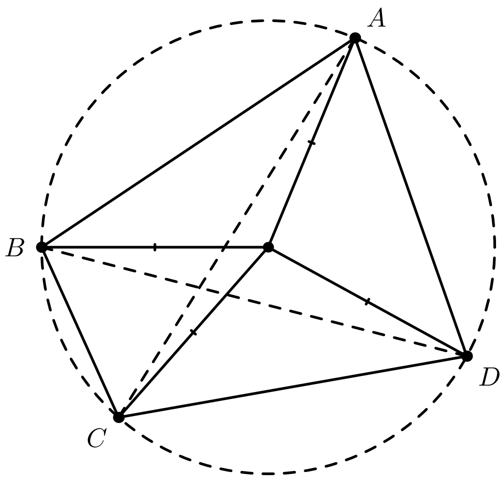


Figure 4: Example 1.10

Example 1.10. Let A, B, C, D, O be points such that $ABCD$ is a convex quadrilateral and $AO = BO = CO = DO$. Show that $\angle ABD = \angle ACD$ and $\angle ABC + \angle ADC = 180°$.

Remarkably, after using 3 isosceles triangles to obtain angle conditions, we arrive with the conclusion that two angles with sides not containing O are congruent. This configuration is a lot more useful than the previous, because we are no longer limited to $60°$ angle or isosceles triangles. However, there are still isosceles triangles hiding in the background.

We formally state it as follows:

Idea 7 (Cyclic Quadrilaterals)**.** *Given convex quadrilateral $ABCD$, the following conditions are equivalent:*

- *$ABCD$ is cyclic*
- *$\angle ABD = \angle ACD$*
- *$\angle ABC + \angle ADC = 180°$*

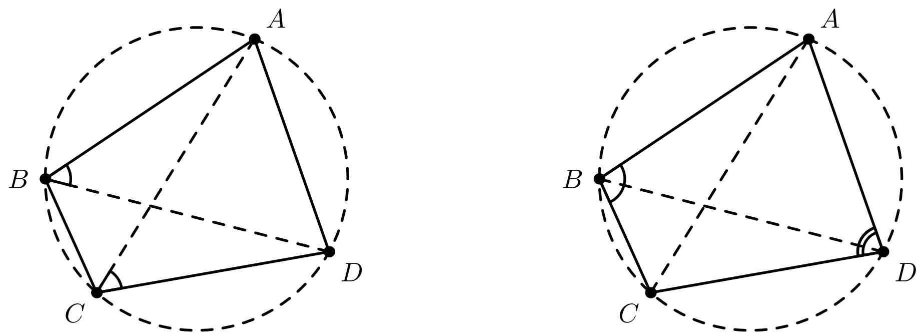


Figure 5: Congruent angles and supplementary angles in a cyclic quadrilateral

This is very useful when there are two arbitrary congruent angles that "cross". Keep in mind that the later two conditions can be written in different variables. Hence, we may do something like $\angle ABD = \angle ACD \iff \angle ADB = \angle ACD$, to "switch" the congruent angle condition over to another pair of angles.

From this equivalence relation, we may also assign angle measurements to arcs, and the above conditions are the same as

- Each arc on a circle has an angle measurement, and angle/arc addition applies normally.
- The total angle measurement of the circle is $360°$.
- Inscribed angles of a circle have measure half that of the arc angle measure.

which are the standard facts of a circle introduced in geometry class. However, often times the circle/center are not drawn, and it is better to think in terms of the quadrilateral (Idea 1).

Remark 1.3. *There is a special case worth noting: Let points A, B, C be on circle ω, if AB is a diameter, then $\angle ACB = 90°$ for any C. The converse is also true. This is known as Thales's theorem.*

With our new tools, we can tackle a series of new problems.

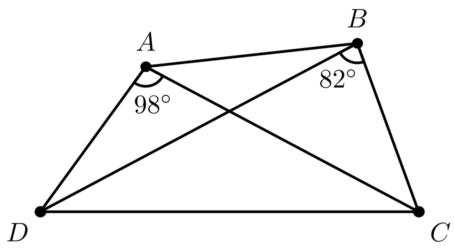
Example 1.11. Convex quadrilateral $ABCD$ satisfies $AC \perp BD$, $\angle ABD = \angle ACD = 65°$, and $\angle ADB = 80°$. Find the measure of $\angle ACB$.

Example 1.12. (2015 PUMaC) Cyclic Quadrilateral $ABCD$ satisfies $\angle ADC = 2 \cdot \angle BAD = 80°$ and $\overline{BC} = \overline{CD}$. Let the angle bisector of $\angle BCD$ met AD at P. What is the measure, in degrees, of $\angle BPD$?

Cyclic quadrilaterals are indeed so powerful that they will continue to pop up in our study of geometry. Specifically for this handout, we will go into more detail of angle chasing and cyclic quadrilaterals in chapters 2 and 5.

Problems

Problem 1.1. (2014 HMMT) In quadrilateral $ABCD$, $\angle DAC = 98°$, $\angle DBC = 82°$, $\angle BCD = 70°$, and BC = AD. Find $\angle ACD$.



Problem 1.2. (Van Schooten's Theorem) Given circle ω and inscribed equilateral $\triangle ABC$, on minor arc AB an arbitrary point P is chosen. Prove that $AP + BP = CP$. **Hint**: 75

Problem 1.3. (2003 AIME I) Triangle ABC is isosceles with $AC = BCC$ and $ACB = 106°$. Point M is in the interior of the triangle so that $\angle MAC = 7°$ and $\angle MCA = 23°$. Find the number of degrees in $\angle CMB$. **Hint**: 13

Problem 1.4. (2016 CMIMC) Suppose $ABCD$ is a convex quadrilateral satisfying $AB = BC, AC = BD$, $\angle ABD = 80°$, and $\angle CBD = 20°$. What is $\angle BCD$ in degrees?

Problem 1.5. (2001 AMC 12) In $\triangle ABC$, $\angle ABC = 45°$. Point D is on BC so that $2 \cdot BD = CD$ and $\angle DAB = 15°$. Find $\angle ACB$. **Hint**: 8

Problem 1.6. (2017 CMIMC) Cyclic quadrilateral $ABCD$ satisfies $\angle ABD = 70°$, $\angle ADB = 50°$, and $BC = CD$. Suppose AB intersects CD at point P, while AD intersects BC at point Q. Compute $\angle APQ - \angle AQP$. **Hint**: 32 22

Problem 1.7. (2017 IMO Shorlist) Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent. **Hint**: 36 4

Problem 1.8. (Langley's Adventitious Angles) Triangle ABC is isosceles with $AC = BC$ and $\angle C = 20°$. D, E are on AC, BC, respectively, such that $\angle ABD = 60°$ and $\angle BAE = 50°$. Find the measure of $\angle EDB$. **Hint**: 56 29

2 Similar Triangles - Part I

Similarity is a powerful notation in geometry to relate arbitrary lengths. In this chapter, we will study similar triangles, and examine the technique of constructing parallel lines. We will also cover useful results like the Angle Bisector's Theorem, Ptolemy's Theorem, and Stewart's Theorem.

This chapter serves as an introduction for identifying similar triangles and constructing parallel lines to complete problems.

Proportionality

What are lengths of segments? Unfortunately, we only have introduced the notation of congruence segments, and while we may add segments up to obtain integer multiples, our options are limited. To deal with the general relation between lengths, we introduce the idea of scaling and similarity.

Idea 8 (Similarity). *Given two triangles $\triangle ABC$ and $\triangle XYZ$, the following conditions are equivalent.*

- *The two triangles $\triangle ABC$ and $\triangle XYZ$ are similar.*
- *(AA) Two of the corresponding angles are congruent - that is, $\angle ABC = \angle XYZ$ and $\angle ACB = \angle XZY$.*
- *(SSS) Three of the corresponding segments are in equal proportion - that is,*

$$\frac{AB}{XY} = \frac{BC}{YZ} = \frac{CA}{ZX}.$$

- *(SAS) Two of the corresponding segments are in equal proportion and the formed angle is congruent - that is,*

$$\frac{AB}{XY} = \frac{BC}{YZ} \quad \text{and} \quad \angle ABC = \angle XYZ.$$

We write $\triangle ABC \sim \triangle XYZ$.

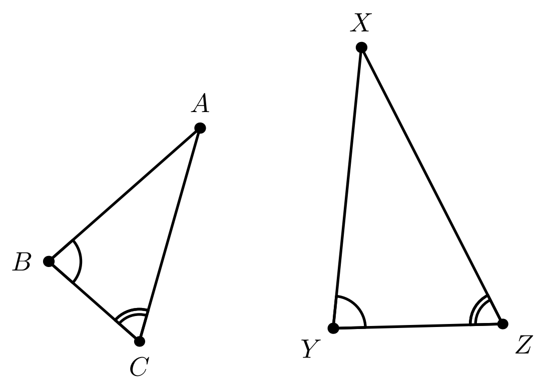


Figure 6: Similar triangles $\triangle ABC$ and $\triangle XYZ$

Now, with the addition of ratios, we are able to solve for many more segment lengths. Beware that algebraic manipulations (Idea 2) are essential to get desired quantities, especially in more complex problems.

The key to these problems is to find parallel lines, and construct more if necessary. Parallel lines automatically create congruent interior angles or vertical angles to yield similar triangles.

Example 2.1. Given segment AB, let points E, F be on the same side of AB such that $AE \perp AB$, $BF \perp AB$. Denote the intersection of EB and AF by I. If $AE = 20$, $BF = 30$, find the distance from I to AB.

Example 2.2. (2016 AMC 10A) In rectangle $ABCD$, $AB = 6$ and $BC = 3$. Point E between B and C, and point F between E and C are such that $BE = EF = FC$. Segments $\overline{AE}$ and $\overline{AF}$ intersect $\overline{BD}$ at P and Q, respectively. The ratio $BP : PQ : QD$ can be written as $r : s : t$ where the greatest common factor of r, s, and t is 1. What is $r + s + t$?

Parallelism

Even if there are no parallel lines initial given, we can still create our own.

Idea 9 (Construction II)**.** *Drawing parallel lines is an effective method to create new meaningful similar triangles.*

There isn't any general rule on when to draw parallel lines. However, it is useful to draw the parallel line through and see if we can find two pairs of similar triangles. Let us illustrate this through two important examples. First is the angle bisector theorem, which is a frequently used result in competition math.

Example 2.3. (Angle Bisector Theorem) In triangle ABC, D is on BC such that $\angle BAD = \angle DAC$. Prove that $\frac{AB}{AC} = \frac{BD}{DC}$. **Hint**: 24 59

For the next problem, we need to be a bit clever. As a remark, a line is "parallel" to itself, in the sense that if there are two segments on the same line, then a third segment will be parallel to both segments.

Example 2.4. (2004 AMC 10B) In $\triangle ABC$ points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $\frac{AT}{DT} = 3$ and $\frac{BT}{ET} = 4$, what is $\frac{CD}{BD}$? **Hint**: 21

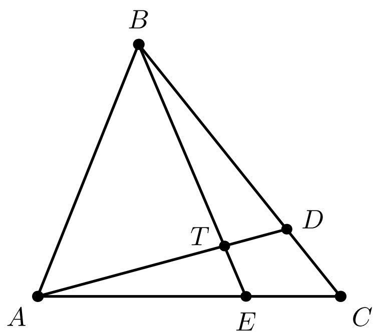


Figure 7: Example 2.4

This parallel segment will be used in many problems involving cevian ratios. We will also dive into more detail of this technique in Chapter 4.

It is worth discussing another specific idea that pops up from time to time.

Idea 10 (Parallelogram Trick)**.** *When a segment is bisected, creating a parallelogram with the diagonal as the bisected segment can enable us to "switch" the condition to the other diagonal being bisected, or other parallel conditions.*

This idea can often be very helpful when finding lengths, such as medians in a triangle, or in any other case where segments are bisected.

Remark 2.1. *This idea is based on the properties of the parallelogram that the diagonals bisect each other, which in turn is enforced by triangle congruence conditions. Again, we see the power of congruence segments.*

Example 2.5. In parallelogram $ABCD$, let M be the midpoint of BC. Define N as the foot of the altitude from A to line MD. Show that $BA = BN$.

Example 2.6. In triangle ABC, let D be on BC such that $BD = CD$. Suppose that points E, F are chosen on AC, AB such that AD, BE, CF are concurrent. Show that $EF \parallel BC$. **Hint**: 38

Circles, Power of a Point and Ptolemey's

We are back with circles. Like how parallel lines create similar triangles, congruent inscribed angles also create similar triangles. Remember the properties of the cyclic quadrilateral (Idea 7) ?

Example 2.7. (Power of a Point) Given point P and circle ω. Suppose that lines m pass through P and intersect ω at A, B. Similarly, have line n pass through P and intersect ω at C, D. Prove that

$$PA \cdot PB = PC \cdot PD.$$

Prove this for both P inside and outside ω. What happens if line m contains a diameter of ω?

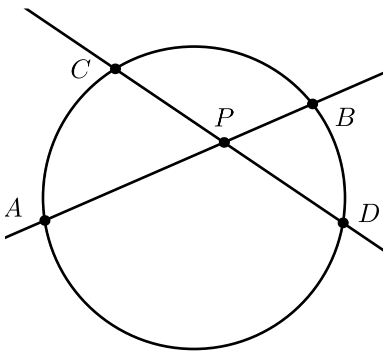


Figure 8: Example 2.7

Power of a point is used very commonly when chord lengths are present in a problem. Do remember that this result is nothing but congruent angles in a cyclic quadrilateral leading to similarity.

Remark 2.2. *It is worth noting that both a cyclic quadrilateral and a parallelogram have two similar triangles formed by the intersecting diagonals, just oppositely oriented! The cyclic quadrilateral often turns out to be more useful, as it ends up with greater degrees of freedom.*

Example 2.8. (1997 AHSME) Triangle ABC and point P in the same plane are given. Point P is equidistant from A and B, angle APB is twice angle ACB, and $\overline{AC}$ intersects $\overline{BP}$ at point D. If $PB = 3$ and $PD = 2$, then $AD \cdot CD =$

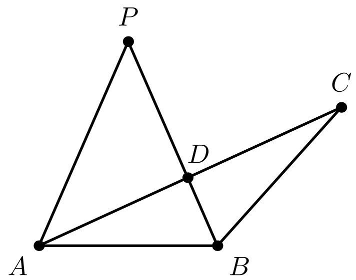


Figure 9: Example 2.8

We may also apply similar triangles to cyclic quadrilaterals, to relate the lengths of the sides and the diagonals. However, we need to be a bit clever in setting up similar triangles.

Example 2.9. (Ptolemy's Theorem) Prove that in a cyclic quadrilateral $ABCD$,

$$AB \cdot CD + BC \cdot DA = AC \cdot BD$$

Hint: 51

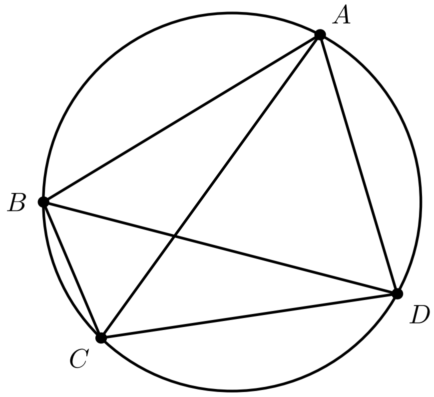


Figure 10: Example 2.9

Example 2.10. (2004 AMC 10B) In triangle ABC we have $AB = 7$, $AC = 8$, $BC = 9$. Point D is on the circumscribed circle of the triangle so that AD bisects angle BAC. What is the value of $\frac{AD}{CD}$?

It may be worthwhile to express the length of a single diagonal in terms of the sides of the cyclic quadrilateral.

Example 2.11. (Stronger form of Ptolemy's) In cyclic quadrilateral $ABCD$, let $AB = a, BC = b, CD = C, DA = d$. Show that

$$AC^2 = \frac{(ac + bd) \cdot (ad + bc)}{(ab + cd)}$$

Now, we are able to chase any length in a cyclic quadrilateral. Though most of times there is a smarter way to solve the problem, there will be problems where bashing the lengths out is necessary or the most straightforward.

Example 2.12. (2012 AMC 12A) Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

Furthermore, even when cyclic quadrilaterals aren't given, we can create them to use Ptolemy's. We end by proving Stewart's Theorem, a formula used very commonly to find lengths of arbitrary cevians. We will state one more almost obvious but commonly used idea.

Idea 11 (Scaling)**.** *Scaling a section of a figure can simplify the problem and/or convert the problem to a known configuration. Let similarity do the work!*

And finally, we hint at an important way of thinking about complex problems. This is a useful method of thinking when dealing with harder problems.

Idea 12 (Extracting Information)**.** *Be familiar with extracting the useful information out of the diagram. What is the important relation between the quantities, and is there any other way to express them?*

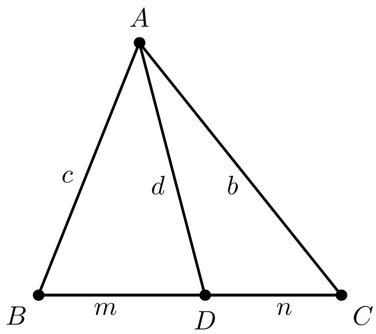


Figure 11: Example 2.12

Example 2.13. (Stewart's Theorem) Let a, b, c be the side lengths of $\triangle ABC$. Let cevian AD have length d, and let $BD = m, CD = n$. Then, show that

$$b^2m + c^2n = d^2a + amn.$$

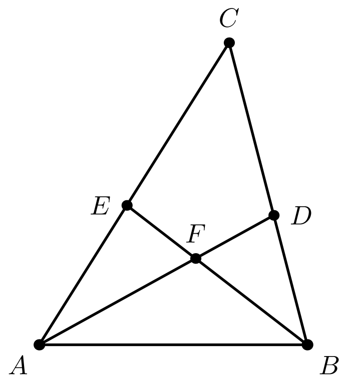
Hint: 34 63

Problem 2.1. Given circle ω and inscribed equilateral $\triangle ABC$, on minor arc AB an arbitrary point P is chosen. CP intersects AB at D. Prove that $\frac{1}{DP} = \frac{1}{BP} + \frac{1}{AP}$.

Problem 2.2. (2009 AIME I) In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$ and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$. **Hint**: 3

Problem 2.3. (2009 AIME I) Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

Problem 2.4. (2016 AMC 12A) In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?



Problem 2.5. (2014 BMT) Consider an isosceles triangle ABC ($AB = BC$). Let D be on BC such that $AD \perp BC$ and O be a circle with diameter BC. Suppose that segment AD intersects circle O at E. If $CA = 2$ what is CE? **Hint**: 70

Problem 2.6. (2017 AMC 12A) Quadrilateral $ABCD$ is inscribed in circle O and has side lengths $AB = 3, BC = 2, CD = 6$, and $DA = 8$. Let X and Y be points on $\overline{BD}$ such that $\frac{DX}{BD} = \frac{1}{4}$ and $\frac{BY}{BD} = \frac{11}{36}$. Let E be the intersection of line AX and the line through Y parallel to $\overline{AD}$. Let F be the intersection of line CX and the line through E parallel to $\overline{AC}$. Let G be the point on circle O other than C that lies on line CX. What is $XF \cdot XG$? **Hint**: 46

Problem 2.7. (2003 USAMO) Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F, while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$. **Hint**: 37 15

3 Right Triangles

In this chapter, we reach the highlight of computational geometry. While similar triangles are great to relate few special lengths in very elegant forms, right triangles enable a wider range of computation. The Pythagorean Theorem is now your best friend. Just drop altitudes and start computing.

In this chapter, we will start with the Pythagorean Theorem and Altitudes in triangles. Through Heron's formula, we motivate the definition of the Area of the triangle. Finally, we examine the circumcircle and incircle, both closely related to right angles and triangle area.

The Pythagorean Theorem

Like isosceles triangles, right triangles are also exceptionally useful. Though isosceles triangles are suited for angle chasing, right triangles are much more powerful with general length chasing, especially when lots of information are given but similar triangles couldn't be used directly.

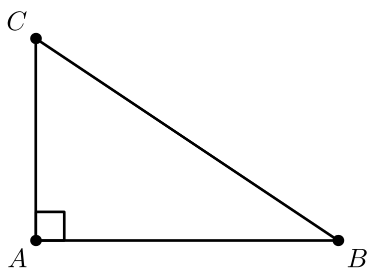


Figure 12: A Right Triangle

Idea 13 (Pythagorean Theorem)**.** *In right triangle ABC with $\angle BAC = 90°$,*

$$AB^2 + AC^2 = BC^2.$$

One nice property of the Pythagorean Theorem is that it is additive with respect to the square of lengths. So, there is no need to solve a quadratic equation for every right triangle of the diagram.

Example 3.1. (2005 AMC 12B) In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

Example 3.2. (2013 HMMT) Let $ABCD$ be an isosceles trapezoid such that $AD = BC$, $AB = 3$ and $CD = 8$. Let E be a point in the plane such that $BC = EC$ and $AE \perp EC$. Compute AE.

When dealing with circles, we often can simplify the problem to right triangles and lines, especially if distances from the center to chords are in the problem. This illustrates an idea used more generally when trying to comprehend a complex problem. It's similar the to previous idea about extracting the key information from a problem statement (Idea 12).

Idea 14 (Simplify)**.** *Reduce complicated figures (circles) to simple, approachable figures (lines).*

Example 3.3. (1983 AIME) A machine-shop cutting tool has the shape of a notched circle, as shown. The radius of the circle is $\sqrt{50}$ cm, the length of AB is 6 cm and that of BC is 2 cm. The angle ABC is a right angle. Find the square of the distance (in centimeters) from B to the center of the circle. **Hint**: 33

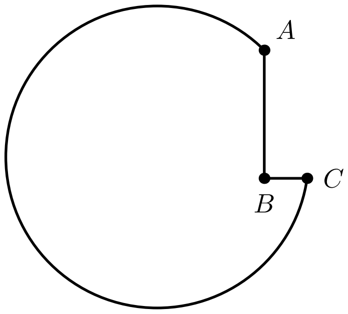


Figure 13: Example 3.3

Example 3.4. (1995 AHSME) Two parallel chords in a circle have lengths 10 and 14, and the distance between them is 6. The chord parallel to these chords and midway between them is of length $\sqrt{a}$ where a is

Question statements involving tangents of circles also tend to involve the use of the Pythagorean Theorem. In these cases, the radii of circles tend to play nicely in the equations.

Idea 15 (Tangents I).

- *For a circle and a tangent line, the line is perpendicular to the radius joining the point of intersection and the circle's center.*
- *The two tangents from a point to the same circle are congruent.*

Idea 16 (Tangents II). *For two tangent circles with a point of contact,*

- *The same line is tangent to both circles at the point of contact.*
- *The point of contact lies on the line joining the two centers.*

We may often simplify the problem by connecting centers of tangent circles and setting their length as $r_1 + r_2$. In the end, problems involving intimidating circles can be reduced to some perpendicular lines in a pattern like a fish bone (Idea 14).

Example 3.5. (2013 SMT) $ABCD$ is a rectangle with $AB = CD = 2$. A circle centered at O is tangent to BC, CD, and AD (and hence has radius 1). Another circle, centered at P , is tangent to circle O at point T and is also tangent to AB and BC. If line AT is tangent to both circles at T , find the radius of circle P. **Hint**: 54

Example 3.6. (2001 AMC 12) A circle centered at A with a radius of 1 and a circle centered at B with a radius of 4 are externally tangent. A third circle is tangent to the first two and to one of their common external tangents as shown. What is the radius of the third circle? **Hint**: 7

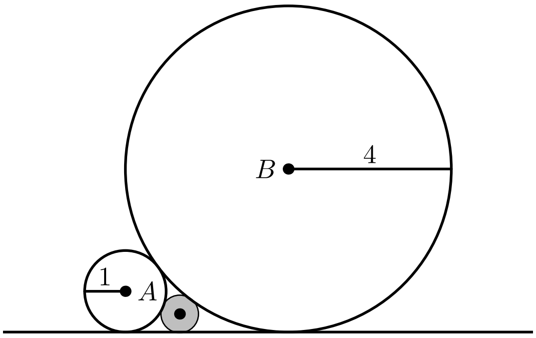


Figure 14: Example 3.6

Heron's formula

We begin with a very logical question involving perpendicular lines (altitudes) in a triangle. All we need is to repeatedly use the Pythagorean Theorem.

Example 3.7. (2014 AMC 10B) Trapezoid $ABCD$ has parallel sides $\overline{AB}$ of length 33 and $\overline{CD}$ of length 21. The other two sides are of lengths 10 and 14. The angles A and B are acute. What is the length of the shorter diagonal of $ABCD$?

Of course, we may generalize this finding to arbitrary triangles. A nice bit of algebra is necessary, though it is clear that our scenario should be solvable (Two unknown distances, two Pythagorean equations).

Example 3.8. (Heron's formula) Show that in triangle ABC, the length of altitude h_a satisfies

$$h_a^2 = \frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4a^2}.$$

Hint: 1 50

Being able to find the altitude in an arbitrary triangle allows for more problems involving length chasing to be solved.

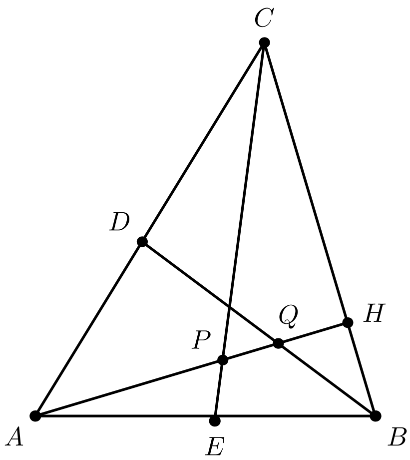


Figure 15: Example 3.9

Example 3.9. (2016 AMC 12B) In $\triangle ABC$ shown in the figure, $AB = 7$, $BC = 8$, $CA = 9$, and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P, respectively. What is PQ?

With the altitude known, we can compute the length for other cevians as well. We will prove Stewart's theorem again with Pythagorean's and Heron's. This is another case where we have so much information that we would be surprised if we couldn't solve for the unknown length!

Idea 17 (Dropping Altitudes)**.** *Even if there are no altitudes or right triangles laid out, if we have enough lengths we can always drop an altitude and start computing. If there are some right angles/important lines, then we can drop even more altitudes parallel/perpendicular to these lines.*

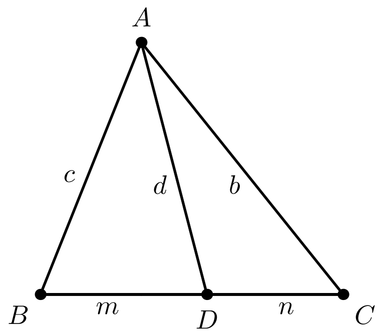


Figure 16: Example 1.10

Example 3.10. (Stewart's Theorem) Let a, b, c be the side lengths of $\triangle ABC$. Let cevian AD have length d, and let $BD = m, CD = n$. Then, show that

$$b^2m + c^2n = d^2a + amn.$$

Area

It appears that we have discovered a really nice, symmetric (well, almost) formula for the altitude of a triangle. In fact, we we considered $a \cdot h_a$, then we would expect this value to be equal to $b \cdot h_b$ or $c \cdot h_c$. Thus, this can be said to be a property of the triangle.

It is not hard to prove, actually without Heron's formula, that $a \cdot h_a = b \cdot h_b$.

Example 3.11. Show with similar triangles that in triangle ABC with altitudes AD and BE, we have $AD \cdot BC = BC \cdot AC$.

This idea is surprisingly powerful that I'm quite surprised doesn't have a name. We chose to define this idea as area, and we typically notate area of $\triangle ABC$ as $[ABC]$. It turns out that area is additive - if we split a triangle into two, then the sum of areas of the two smaller triangles is equal to the area of the larger triangle.

Also, we can extend the notion of area to other polygons as a sum or difference of triangles. We won't be proving this, but the area is well-defined for simple polygons.

Idea 18 (Area of a Polygon)**.** *The area of a polygon satisfies the following properties*

- *The area of a triangle is a side length multiplied by the length of the altitude to that side, divided by 2.* ($A = \frac{1}{2} \cdot b \cdot h$)
- *Area is additive. When two polygons share a side, then the sum of area of individual polygons is the same as the area of the polygon 'merged' from the two polygons removing the side.*

We need not think of area as anything special, but rather just a "nice" property of triangles that allows for creating problems. We may treat this just another length condition. Other times, we can use area ourselves to describe similar triangles.

Remark 3.1. *Unfortunately, we won't quite be able to see what the area of a circle is. This will be inevitable, as the first formal definition of area will probably come from your calculus class. Regardless, you probably know* $A_{circle} = \pi r^2$ *already to deal with problems.*

Inradius and Circumradius

Closely related to right triangles and thus area are the incenter (inradius r) and circumcemter (circumradius R) of a triangle.

Remember that the incenter is equidistant to the sides of the triangle and the circumcemter is equidistant to the points of a triangle. We know that they exist from **Example 1.1**.

Example 3.12. Use similar triangles and ratios to show that in $\triangle ABC$ with circumradius R, we have

$$[ABC] = \frac{abc}{4R}.$$

Hint: 30

The case with the incenter is slightly more complicated, but keep everything of the chapter in mind, and the example can be solved. We will highlight one more idea that is commonly used as a conjunction of similarity and right triangles

Idea 19 (Trigonometry)**.** *When two right triangles are have some related angle condition (such as one angle being double the angle in another), scaling the two right triangles to share a base can simplify the problem.*

The essence of trigonometry is that trigonometry (Idea 19) is just scaling (Idea 11) for right triangles!

Example 3.13. Use similar triangles and ratios to show that in $\triangle ABC$,

$$[ABC] = rs \quad \text{where } s = \frac{a+b+c}{2}.$$

Then, use the idea that area is additive to reprove the statement. **Hint**: 44 11

It turns out that the notation s for the "semi-perimeter" is pretty nice and can be substituted back into Heron's formula. We now combine all our formulae for the area of a triangle.

Idea 20. *Area Formulae Finding different ways of expressing area is the key to many problems. In particular, for a triangle we have*

$$[ABC] = \frac{a \cdot h_a}{2} = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R}$$

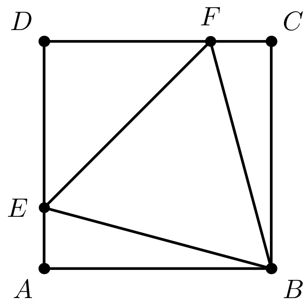
Now, looking in the opposite direction, we finally have altitudes, circumradii and inradii written in terms of side lengths of a triangle. Combining it with a bit more Pythagorean theorem (do keep in mind that everything is doable because of the Pythagorean Theorem), we can compute some interesting lengths in problems.

Example 3.14. (2012 AMC 12A) Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I be the intersection of the internal angle bisectors of $\triangle ABC$. What is BI?

Example 3.15. (2003 AIME II) Find the area of rhombus $ABCD$ given that the radii of the circles circumscribed around triangles ABD and ACD are 12.5 and 25, respectively.

Problem 3.1. (2002 AMC 10A) In trapezoid $ABCD$ with bases AB and CD, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. The area of $ABCD$ is

Problem 3.2. (2004 AMC 10A) Points E and F are located on square $ABCD$ so that $\triangle BEF$ is equilateral. What is the ratio of the area of $\triangle DEF$ to that of $\triangle ABE$?



Problem 3.3. (2009 AMC 12A) A circle with center C is tangent to the positive x and y-axes and externally tangent to the circle centered at $(3, 0)$ with radius 1. What is the sum of all possible radii of the circle with center C?

Problem 3.4. (2008 AIME II) In triangle ABC, $AB = AC = 100$, and $BC = 56$. Circle P has radius 16 and is tangent to $\overline{AC}$ and $\overline{BC}$. Circle Q is externally tangent to P and is tangent to $\overline{AB}$ and $\overline{BC}$. No point of circle Q lies outside of $\triangle ABC$. The radius of circle Q can be expressed in the form $m - n\sqrt{k}$, where m, n, and k are positive integers and k is the product of distinct primes. Find $m + nk$. **Hint**: 26 64

Problem 3.5. (2016 AMC 12A) Circles with centers P, Q and R, having radii $1, 2$ and 3, respectively, lie on the same side of line l and are tangent to l at P', Q' and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of triangle PQR? **Hint**: 57

Problem 3.6. (2011 AIME II) A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be represented as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m + n$ is divided by 1000. **Hint**: 47

Problem 3.7. (2007 AIME II) Four circles ω, ω_A, ω_B, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are 13, 14, and 15, the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. **Hint**: 71 27

Problem 3.8. (2012 AIME II) Triangle ABC is inscribed in circle ω with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle A meets side $\overline{BC}$ at D and circle ω at a second point E. Let γ be the circle with diameter $\overline{DE}$. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. **Hint**: 55 62

4 Similar Triangles - Part II

As we have seen in the previous chapter, the area of the triangle is a very special property that is based on similarity of right triangles. This chapter combines the two ideas from previous chapters: similarity and area.

We will start by examining the relation of triangle areas without altitudes drawn in. Then, we will examine ratios of cevians in a triangle, including the famous Menelaus's and Ceva's theorem. We will also gain familiarity with ratios of areas, and learn to convert between length ratios and area ratios.

Right triangles and Area ratios

For two triangles sharing just one congruent angle, we can still obtain the area ratio of the triangles in terms of the pair of sides, even though there are no right angles labeled anywhere. Try drawing in parallel altitudes, and see what happens.

Example 4.1. (2005 AMC 10A) In ABC we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on AB and AC respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$? **Hint**: 72

Example 4.2. (2004 AMC 10B) In the right triangle $\triangle ACE$, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of $\triangle DBF$ to that of $\triangle ACE$?

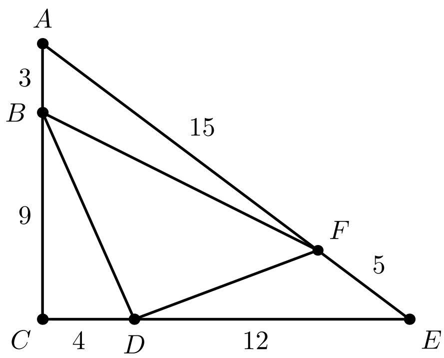


Figure 17: Example 4.2

Cevian Ratios, Menelaus's

Triangles and its cevians are the some of the most common geometry object used for computational geometry problems. It is within our interest to analyze them in greater detail. Though we have done many related problems in chapter 2, we want to form some generalized relations about ratios in a triangle.

For convenience, in this section our reference triangle will be $\triangle ABC$, with D, E, F on BC, CA, AB, respectively, such that the three cevians AD, BE, CF concur at a single point P. (see figure)

Based on our experiences with construction (Idea 3 and Idea 9), we now will write out:

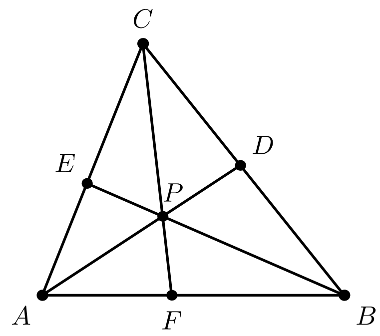


Figure 18: Standard Setup

Idea 21 (Construction III)**.** *When two geometric objects are related by some equivalence relation (congruence or parallelism), but we can not directly use that equivalence, then we should construct a third object equivalent to both.*

This is a very important concept, that was even mentioned in Euclid's axioms, as "Things which are equal to the same thing are also equal to one another."

Indeed, for both congruence construction and parallel line constructions, the way that we actually use that information is to get more expressions than we put in. Ideally, if we draw one segment, and get two relations about that segment, then we have found something new.

For this section, upon construction of parallel lines, we want to express the new segment's length using two different similar triangles. Let us start with a scenario similar to some we encountered in section 2.

Example 4.3. (Menelaus's theorem) In $\triangle ABC$, cevians AD and BE intersect at P. Show that

$$\frac{CE}{EA} \cdot \frac{AP}{PD} \cdot \frac{DB}{BC} = 1$$

Hint: 16

The formula might look a bit scary, but one can always check by the parallel line method. Furthermore, all the letters are "cyclic", which is an aid in writing the correct formula.

Great! Now, we can relate between cevian ratios and side ratios.

Remark 4.1. *Though we stated the ratios of cevians inside a triangle, as more computational problems generally are set up this way, this theorem traditionally stated in terms of ACD as the reference triangle. In this form, it is more convenient to use the converse of the theorem to prove that the three points B, P, E are collinear.*

Unfortunately, a nice formula relating two cevian ratios and one side ratios AP/DP, BP/EP, AE/CE (like what we had in **Example 2.4**) does not exist. Regardless, we can still construct the same parallel line and write the similarity in terms of the given ratios.

Another cute result is the adding ratios result. This time, a bit more algebraic manipulation is necessary. Nonetheless, after constructing the right parallel line, the rest of the work is quite straightforward.

Example 4.4. Show that

$$\frac{AE}{EC} + \frac{BD}{DC} = \frac{CP}{PF}.$$

Finally, we will do an example to see Menelaus's in action.

Example 4.5. (2002 AIME II) In triangle ABC, point D is on $\overline{BC}$ with $CD = 2$ and $DB = 5$, point E is on $\overline{AC}$ with $CE = 1$ and $EA = 3$, $AB = 8$, and $\overline{AD}$ and $\overline{BE}$ intersect at P. Points Q and R lie on $\overline{AB}$ so that $\overline{PQ}$ is parallel to $\overline{CA}$ and $\overline{PR}$ is parallel to $\overline{CB}$. It is given that the ratio of the area of triangle PQR to the area of triangle ABC is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Area Ratios, Ceva's

Now, we turn our view to relations involving three side ratios and three cevian ratios. Though the next two results can be solved algebraically using the two previous results (you are encouraged to check so!), we will utilize a new technique involving area.

Idea 22 (Switching Ratios)**.** *A way to simplify arithmetic about ratios is to convert between length ratios and area ratios, and use the additive property of areas and lengths, respectively.*

We have already seen that area ratios can be written as length ratios, in some cases described in **Example 4.1**. Here, we will see some more ratio breakdowns.

Idea 23 (Areas in a Triangle)**.** *In our standard setup, let $K_a = [BCP]$, and define K_b, K_c, respectively. Express desired ratios in terms of K_a, K_b, K_c,*

This might appear a bit random at first, but it should make more sense as we do work through the examples.

Example 4.6. (Ceva's Theorem) In triangle ABC, let cevians AD, BE, CF intersect at P. Show that

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Hint: 5

Indeed, this is quite faster to derive. However, it is important to bear in mind why labeling K works. Unlike the previous examples involving the creation of solely parallel lines, the area solution is secretly creating perpendicular lines!

We drop the perpendiculars from two vertices A, B, to line CD, and create similar right triangles. But the ratios are able to "switch over" to the other four altitudes, by the properties of the triangle area (**Example 3.11**).

Let us use the area idea on two more examples.

Example 4.7. (1992 AIME) In triangle ABC, A', B', and C' are on the sides BC, AC, and AB, respectively. Given that AA', BB', and CC' are concurrent at the point O, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$. **Hint**: 43

Example 4.8. (2018 PUMaC) Let $\triangle ABC$ be triangle with side lengths $AB = 9, BC = 10, CA = 11$. Let O be the circumcenter of $\triangle ABC$. Denote $D = AO \cap BC, E = BO \cap CA, F = CO \cap AB$. If $1/AD + 1/BE + 1/FC$ can be written in simplest form as $\frac{a\sqrt{b}}{c}$, find $a = b + c$.

Remark 4.2. *Other then defining K above (Idea 23), there are also other ways to set up 3 special values on a triangle cevian system. A notable example is mass points, which is related to the adding ratios result.*

When Equality Occurs

The fact that the area is two lengths multiplied together means that we can decompose area ratios to ratios of pairs of lengths. It is quite common in problems for triangles to share the same height, and have different bases on the same line. Here, the ratio of heights vanish. Then, the area ratios is just the ratio of base length. We formally write this idea as follows:

Idea 24 (Area Ratios)**.**

- *If two triangles have an equal pair of base lengths, then the ratio of their areas is equal to the ratio of their heights.*
- *If two triangles have an equal pair of heights, then the ratio of their areas is equal to the ratio of their base lengths.*

Example 4.9. (2008 AMC 10A) Trapezoid $ABCD$ has bases $\overline{AB}$ and $\overline{CD}$ and diagonals intersecting at K. Suppose that $AB = 9$, $DC = 12$, and the area of $\triangle AKD$ is 24. What is the area of trapezoid $ABCD$?

Example 4.10. (2006 AMC 10B) A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown. What is the area of the shaded quadrilateral? **Hint**: 39 66

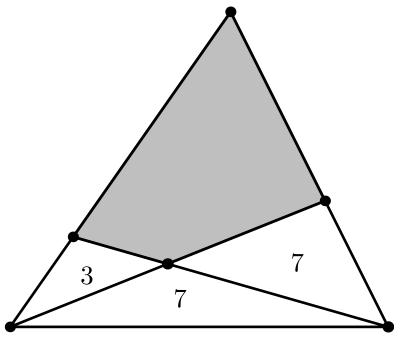


Figure 19: Example 4.10

There are finally a class of problems that require to prove equal areas. Usually, using area ratios are not the quickest and most direct forms of proof. However, they are probably among the most elegant forms of proof, and can often be transcribed into proofs without words. In other problems, using equal areas can change the problem statement to simplify the remainder of the problem.

Our central idea this section can be visualized as follows:

Idea 25 (Sliding the Third Point)**.** *Given $\triangle ABC$, let ℓ be the line through A parallel to BC. Then, for any point A' on ℓ,*

$$[ABC] = [A'BC].$$

Example 4.11. (Euclid's proof) Given $\triangle ABC$ with hypotenuse AC, let points U, V, W, X, Y, Z be points such that $ABVU$, $BCXW$, $ACZY$ are squares outside $\triangle ABC$. Using area ratios, prove that

$$[ABVU] + [BCXW] = [ACZY].$$

Hint: 41 67

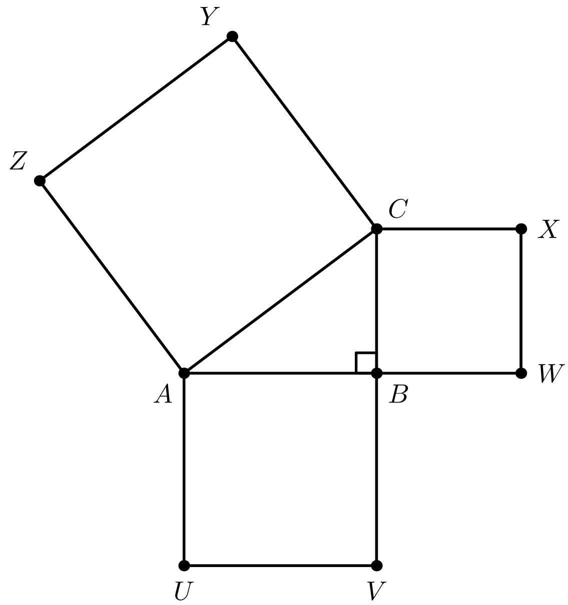


Figure 20: Example 4.11

Example 4.12. (2014 PUMaC) $\triangle ABC$ has side lengths $AB = 15, BC = 34$, and $CA = 35$. Let the circumcenter of ABC be O. Let D be the foot of perpendicular from C to AB. Let R be the foot of perpendicular from D to AC, and let W be the perpendicular foot from D to BC. Find the area of quadrilateral $CROW$. **Hint**: 6 18

Problem 4.1. (2013 AMC 10B) In triangle ABC, medians AD and CE intersect at P, $PE = 1.5$, $PD = 2$, and $DE = 2.5$. What is the area of $AEDC$?

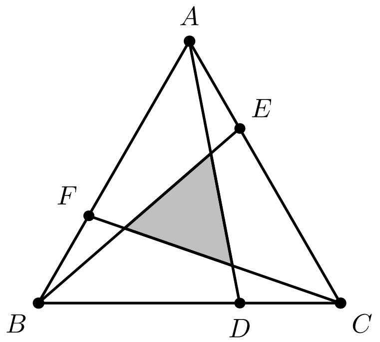
Problem 4.2. (2020 SMT) Let ABC be an acute triangle with $BC = 4$ and $AC = 5$. Let D be the midpoint of BC, E be the foot of the altitude from B to AC, and F be the intersection of the angle bisector of $\angle BCA$ with segment AB. Given that AD, BE, and CF meet at a single point P, compute the area of triangle ABC. Express your answer as a common fraction in simplest radical form.

Problem 4.3. (2018 HMMT) In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

Problem 4.4. (2017 AIME II) Rectangle $ABCD$ has side lengths $AB = 84$ and $AD = 42$. Point M is the midpoint of $\overline{AD}$, point N is the trisection point of $\overline{AB}$ closer to A, and point O is the intersection of $\overline{CM}$ and $\overline{DN}$. Point P lies on the quadrilateral $BCON$, and $\overline{BP}$ bisects the area of $BCON$. Find the area of $\triangle CDP$. **Hint**: 35

Problem 4.5. (2019 HMMT) Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$. **Hint**: 45

Problem 4.6. (2003 AIME I) In $\triangle ABC, AB = 360, BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects angle ABC. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. The ratio $DE : EF$ can be written in the form m/n, where m and n are relatively prime positive integers. Find $m + n$.
Hint: 76



Problem 4.7. (One-seventh Area Triangle) In triangle ABC, points D, E, F are on segments BC, CA, AB, respectively, such that $BD = 2DC$, $CE = 2EA$, $AF = 2FB$. Prove that the triangular region bounded by AD, BE, CF has one-seventh the area of triangle ABC.

- In the case that $\triangle ABC$ is equilateral, prove the statement without writing out cevian ratios (no Menelaus's)
- In the general case, reprove the statement.

Hint: 58

5 Circles

For the final chapter, we will go over some challenging problems, some of which are intended to be beyond the level of problems to solve on competitions. This chapter contains some specific configurations, which we will tie in with the rest of the handout. Circles are a main theme, and synthetic techniques from previous chapters are frequently used.

In this chapter, you are on your own. Examples and problems will be grouped for you, but it is your own job to discover the main idea behind these problems.

Don't be afraid to spend lots of time on a single problem. Due to the difficulty of these problems, hints are frequently provided, and you are strongly encouraged to use them. Good luck.

Recap

We will summarize some key synthetic ideas of the handout.

- Look for congruence and symmetry. Create congruent triangles. (Idea 3)
- Use cyclic quadrilaterals to find angle relations (Idea 7)
- Construct parallel lines and parallelograms. (Idea 9, Idea 10)
- Similar triangles is the most powerful technique to relate lengths. (Idea 8)

Warm Up: Angle Chasing

Example 5.1. (2015 AIME I) Point A, B, C, D, and E are equally spaced on a minor arc of a circle. Points E, F, G, H, I and A are equally spaced on a minor arc of a second circle with center C as shown in the figure below. The angle $\angle ABD$ exceeds $\angle AHG$ by $12°$. Find the degree measure of $\angle BAG$. **Hint**: 25 10

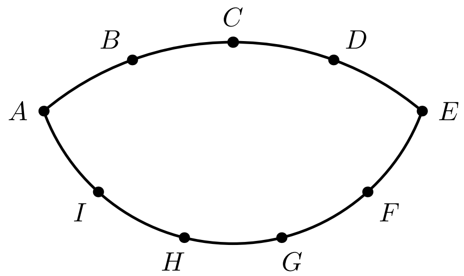


Figure 21: Example 5.1

Example 5.2. (2012 HMMT) There are circles ω_1 and ω_2. They intersect in two points, one of which is the point A. B lies on ω_1 such that AB is tangent to ω_2. The tangent to ω_1 at B intersects ω_2 at C and D, where D is the closer to B. AD intersects ω_1 again at E. If $BD = 3$ and $CD = 13$, find EB/ED. **Hint**: 60 40

Example 5.3. (2019 AIME I) In convex quadrilateral $KLMN$, side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO. **Hint**: 69

Symmetry

Example 5.4. (2007 AIME II) Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.

Example 5.5. (2005 AIME II) Square $ABCD$ has center O, $AB = 900$, E and F are on AB with $AE < BF$ and E between A and $F, m\angle EOF = 45°$, and $EF = 400$. Given that $BF = p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$. **Hint**: 2

Example 5.6. (2018 AMC 12B) Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$? **Hint**: 42 65

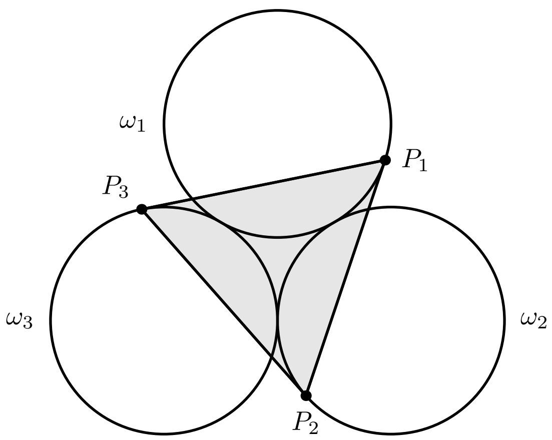


Figure 22: Example 5.6

Cyclic Quadrilaterals and Similarity

Example 5.7. (Existence of Orthocenter) Show that in a triangle ABC, the three altitudes AD, BE, CF concur at one point. **Hint**: 19 31

Example 5.8. (2013 AMC 10B) In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$? **Hint**: 23

Example 5.9. (2010 IMO Shortlist) Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$. **Hint**: 52 49

Reflections and Bisections

Example 5.10. (2019 HMMT) Isosceles triangle ABC with $AB = AC$ is inscribed in a unit circle Ω with center O. Point D is the reflection of C across AB. Given that $DO = \sqrt{3}$, find the area of triangle ABC.

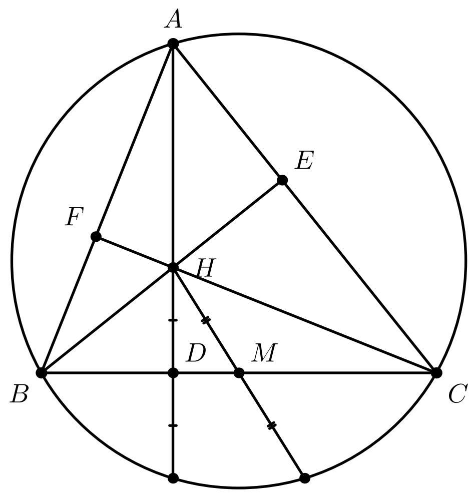


Figure 23: Example 5.11

Example 5.11. (Reflection of Orthocenter) In triangle ABC, let H be the orthocenter, D be the foot of altitude from A to BC, and M be the midpoint of BC. Show that the reflection of D over BC and the reflection of D over M lies on the circumcircle of $\triangle ABC$. **Hint**: 9 53

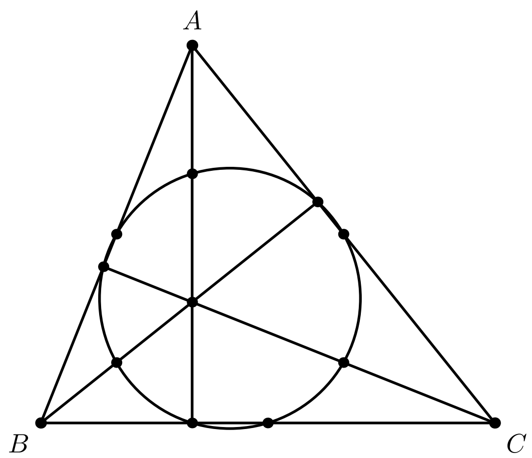


Figure 24: Example 5.12

Example 5.12. (9-point Circle) In $\triangle ABC$, show that the midpoints of the sides, the foots of altitudes, and the midpoints between the orthocenter and the vertices all concur on one circle. **Hint**: 28 14

Congruence and Overlay

Example 5.13. (2016 AIME I) In $\triangle ABC$ let I be the center of the inscribed circle, and let the bisector of $\angle ACB$ intersect AB at L. The line through C and L intersects the circumscribed circle of $\triangle ABC$ at the two points C and D. If $LI = 2$ and $LD = 3$, then $IC = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$. **Hint**: 17

Example 5.14. (2013 HMMT) Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD. **Hint**: 12

Example 5.15. (2005 IMO Shortlist) Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle. **Hint**: 61 48

Example 5.16. (Euler's formula) In $\triangle ABC$, let O, I denote the circumcenter, incenter, respectively. Show that

$$OI^2 = R(R - 2r)$$

Hint: 74

Final Advice

To approach difficult problems, we need to find what really is happening in the problem.

- Simplify the problem. Have we seen this configuration before? (Idea 14)
- Determine what really is the relationship between parts of the configuration. (Idea 12)
- Move parts of the configuration around, and change the problem statement. (Idea 3, Idea 12)

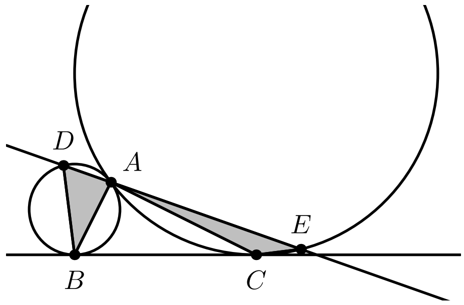
Draw big, labeled diagrams. Redraw diagrams after changing the problem statement, or when viewing the problem from a different prospective.

At this point, I highly recommend that you have worked through all previous examples and problems before proceeding. In particular, make sure to really understand **Example 5.2**, **Example 5.7**, and **Example 5.11**.

When you are ready, flip over the page.

Challenging problems

Problem 5.1. (2015 AIME II) Circles $\mathcal{P}$ and $\mathcal{Q}$ have radii 1 and 4, respectively, and are externally tangent at point A. Point B is on $\mathcal{P}$ and point C is on $\mathcal{Q}$ so that line BC is a common external tangent of the two circles. A line ℓ through A intersects $\mathcal{P}$ again at D and intersects $\mathcal{Q}$ again at E. Points B and C lie on the same side of ℓ, and the areas of $\triangle DBA$ and $\triangle ACE$ are equal. This common area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.



Problem 5.2. (2019 AIME I) Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 5.3. (2019 AIME II) In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Problem 5.4. (2020 AIME I) Let $\triangle ABC$ be an acute triangle with circumcircle ω, and let H be the intersection of the altitudes of $\triangle ABC$. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3, HX = 2$, and $HY = 6$. The area of $\triangle ABC$ can be written in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

A Problem Sources

The examples and problems used in this handout comes from the following competitions:

- (AMC) American Mathematics Competitions
- (AHSME) American High School Mathematics Examination
- (AIME) American Invitational Mathematics Examination
- (USAMO) United States of America Mathematical Olympiad
- (IMO Shortlist) Shortlist of the International Mathematical Olympiad
- (HMMT) Harvard-MIT Mathematics Tournament
- (PUMaC) Princeton University Mathematics Competition
- (CMIMC) Carnegie Mellon Informatics and Mathematics Competition
- (SMT) Stanford Mathematics Tournament
- (BMT) Berkeley Math Tournament

B Hints

1. Let D be the foot of altitude h_a. Then, $CD = x, BD = a - x$. Can we solve x?
2. Two 45° makes a 90° angle. How can we use the symmetry of a square?
3. Can we scale (Idea 11) to turn the parallelogram into a triangle?
4. Let AB and CD meet at Z. Why is I, the incenter of BZC, important?
5. Try writing AF/FB in terms of K's given in Idea 23
6. The circumcenter lies on the intersection of perpendicular bisectors. Let X, Y be midpoints of AC, BC, respectively.
7. BT is tangent to both circles. How can we use the property of tangents (Idea 15)?
8. The condition $2 \cdot BD = CD$ is not helpful. Can we get congruent segments instead?
9. I see three congruent triangles.
10. Look at $\triangle ACE$ and in particular $\angle ACE$.
11. Let AD be an altitude, and have CI intersect AD at P. Can we find PD? How can we then use Idea 19?
12. Let X be the point of contact between the incircle and AB. What also has length AX?
13. Try Idea 6. Do you see multiple congruent figures?
14. Let N be the midpoint of H, O.
15. What angles are congruent to $\angle MBC$?
16. Draw in that line segment we need from Idea 21.
17. Try to prove $AD = BD = ID$.
18. Split $CROW$ up cleverly. Then, try to find other triangles of equal area (Idea 25)!
19. First, let H be the intersection of BE, CF. There are two cyclic quadrilaterals.
20. Construct D so $\triangle ADC$ is isosceles. Can you now apply Idea 6?
21. There is one convenient line segment parallel to both CD and BD.
22. Why is C special (Idea 4)? Are there any hidden isosceles triangles?
23. Various lengths can be computed independent of each other.
24. Draw line ℓ through C parallel to AB (Idea 9).
25. Remember (Idea 7), and also measures of arcs.
26. First, we simplify the problem to a fishbone by Idea 14.
27. Let $\triangle O_A O_B O_C \sim \triangle ABC$. Express the ratio in two ways: One involving the circumraduis, and one involving the angle bisectors and Idea 19.
28. What characteristic do all 9 points share?
29. Note that $\triangle CDB$ is isosceles. Now, reflect Y over the line of symmetry for $\triangle CDB$ (Idea 6).
30. Let M be the midpoint of BC. Find a pair of similar triangles.
31. Find angles congruent to $\angle HEF$, and also express it in terms of angles in $\triangle ABC$.
32. We like equilateral triangles. Let us take R on AQ so $\triangle APR$ is equilateral.

33. Extend parallel and perpendicular segments, until we can apply the Pythagorean Theorem (Idea 13).
34. Look at $\triangle ADB$ and $\triangle ADC$ separately. What special relations (Idea 12) exist between the two triangles?
35. Complementary counting.
36. I see three congruent triangles (Idea 3).
37. The equation $MB \cdot MD = MC^2$ seems to resemble ratios (Idea 8).
38. Rather than angles, use length ratios.
39. Try splitting the region into two.
40. Honestly, just look at the solution.
41. Let Q be foot from B to ZY. Which figures have area?
42. Draw P_1' on its location relative to ω_1, but instead on ω_2, What can we say by symmetry?
43. Once again, write AO/OA' in terms of K's given in Idea 23. Don't be afraid of algebra (Idea 2)!
44. If we have an angle bisector, we should somehow be able to use the angle bisector theorem.
45. Reduce the parallelograms to triangles ACD, ABD (Idea 12). What must be true of BC?
46. Just follow through, and unwind the mess; use similarity (Idea 8) to change the final answer into something nicer, one step at a time.
47. Find the cyclic quadrilateral.
48. Remember, $\angle CEI = 90^\circ$.
49. Remember that $\angle C = \angle AFE = \angle BFD$.
50. Don't be afraid of algebra (Idea 2)!
51. Reflect AC over the angle bisector of $\angle BAD$. What does this imply?
52. There is a cyclic quadrilateral containing Q.
53. Let the reflection of H over BC, M be $H', H^\dagger$, respectively. For the former, use angles; for the later, use lengths.
54. AT is tangent to both circles. How can we use the property of tangents (Idea 15)?
55. Consider GE, the diameter of ω.
56. Draw X so that XDB is equilateral (Idea 5). Draw Y as X reflected over line of symmetry for isosceles $\triangle ACB$ (Idea 6).
57. Parallel and perpendicular lines.
58. Let AD, BE intersect at X. Look at $\triangle BXC$.
59. Isosceles triangles (Idea 2) can convert angle and length congruence.
60. Can we still use angles of cyclic quadrilaterals (Idea 7) for tangent lines? Yes, we can.
61. Let CI intersect the circumcircle of $\triangle ABC$ at P. What also has length CE
62. Can we find AD, GD, and GE separately?
63. Try rotating $\triangle ABD$ at an angle of $\angle DAC$ with respect to A. Use scaling (Idea 11) as appropriate.

64. Then, apply Idea 19 to relate the lengths we are missing.
65. $O_1P_1P_1'O_2$ is a parallelogram. Furthermore, $\angle P_2O_2P_1' = 120°$.
66. For each length ratio, find two corresponding area ratios!
67. $\triangle ZAB \cong \triangle CAU$.
68. This time, what happens if we reflect E over BD? How may we use Idea 5?
69. Find three congruent angles.
70. Where does circle O intersect AC?
71. Idea 8 does the trick. What can I say about O with respect to $\triangle O_AO_BO_C$?
72. Just drop two altitudes onto AB (Idea 17). Any similar triangles?
73. Try rotating P $60°$ about A (Idea 5). Do you see any isosceles triangles and congruent triangles?
74. Rewrite the formula with Power of a Point.
75. Rotate P $60°$ about C. Do you see any equilateral triangles?
76. Draw D' so that $BD \parallel D'C$, and $BDCD'$ is an isosceles trapezoid. What have we really done here?