Ideas and Insight in Synthetic Geometry: Solution to Examples

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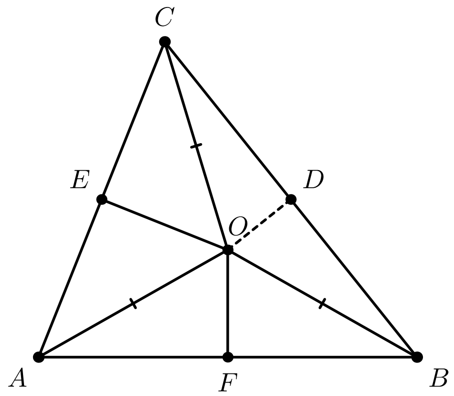
1 Solutions to Section 1 Examples

Solution to Example 1.1.

To show concurrency, our strategy is to define the point of intersection of two of the three lines. Then, we argue that the point must lie on the third line.

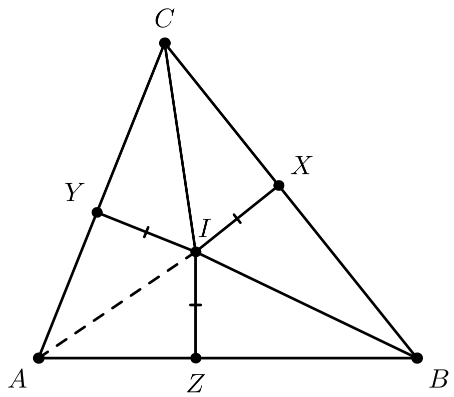
We first show that the three perpendicular bisectors concur at one point. Let the midpoints of BC, CA, AB be D, E, F, respectively.

Suppose that perpendicular bisector to AB, AC intersects at O. We know that $AE = EC$, $EO = EO$, and $\angle AEO = \angle CEO = 90°$. So, by SAS/HL congruence we have that and $\triangle AOE \cong \triangle COE$. A similar argument shows $\triangle AOF \cong \triangle BOF$. Therefore, we have $AO = OB$ and $AO = CO$.



So, we must also have $OB = OC$, or O is equidistant from B and C. Since $\triangle COB$ is isosceles, we also have $\angle OCD = \angle OBD$. Furthurmore, we started with $CD = DB$. By SAS congruence, $\triangle OCD = \triangle OBD$. Thus, $\angle ODC = \angle ODB = 90°$, and O also lies on the perpendicular bisector of CB.

Now, let angle bisectors for $\angle B$ and $\angle C$ intersect at I. Since BI is an angle bisector of B, I must be equidistant from AB and AC. Let the perpendiculars from I to AB, BC be Z, X, respectively. Then, $IZ = IX$.



By similar reasoning on $\angle C$, $IX = IY$. Then, $IY = IZ$, which implies by HL congruence that $\triangle IYA = \triangle IZA$, and I again must lie on the angle bisector of $\angle A$. So, the three angle bisectors are concurrent.

Solution to Example 1.2.

This problem is straightforward computation. However, it is worth noting a small "formula" (though you really will be using this so often it just comes naturally):

Suppose that the angles in an isosceles triangle are x, x, y. Then, $x + x + y = 180°$. We may also write

$$y = 180° - 2x \text{ or } x = \frac{180° - y}{2}$$

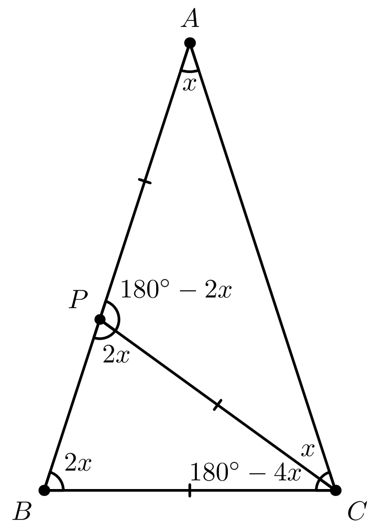
Returning to our example, we may compute

$$\angle BAC = \frac{180° - 40°}{2} = 70° \quad \text{and} \quad \angle DAC = \frac{180° - 140°}{2} = 20°.$$

Finally, since D is in the interior of $\angle BAC$, by angle addition (subtraction) we have

$$\angle BAD = \angle BAC - \angle DAC = 70° - 20° = 50°.$$

Solution to Example 1.3.



This time, there aren't any angles that we can directly obtain. Instead, we will need to solve for using algebra.

Let $\angle BAC = x$. Then, we fill in angles of $\triangle APC$, keeping in mind that $AP = PC$.

$$\angle APC = 180 - 2x \quad \text{and} \quad \angle PAC = x$$

Next, we know that $\angle BPC = 180 - \angle APC = 2x$. So, looking at isosceles $\triangle BCP$,

$$\angle PBC = 2x \quad \text{and} \quad \angle PCB = 180 - 4x$$

We still haven't used that $\triangle BAC$ is isosceles, or $\angle ABC = \angle ACB$. Then,

$$2x = 180 - 4x + x \implies x = 36°$$

In these problems, it is efficient to do the algebra on the diagram, by simply labeling in known angles as you progress through the algebra.

Solution to Example 1.4.

Let $\angle ACB = x$, and $\angle ABC = x - 30°$. It follows that

$$\angle BAC = 180° - x - (x - 30°) = 210° - 2x.$$

However, $\triangle BAD$ is isosceles, so

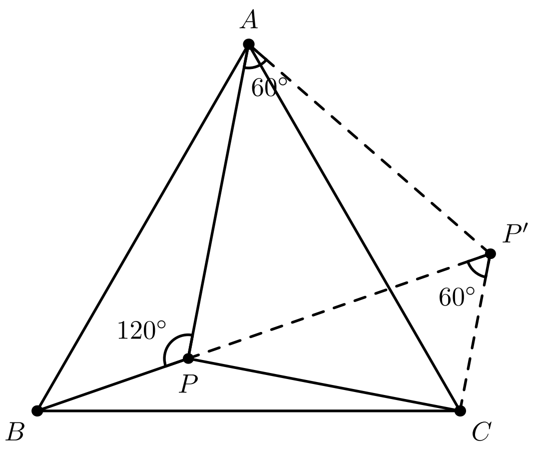
$$\angle ABD = \frac{180 - \angle BAD}{2} = x - 15°$$

As $\angle CBD + \angle ABC = \angle ABD$, we finally get

$$\angle CBD + x - 30° = x - 15° \implies \angle CBD = 15°.$$

Nicely, x cancels, and we get a definitive solution. It turns out that $\angle ACB$ and related angles are indeed not well-defined, but $\angle CBD$ somehow is. Indeed, this example may just attributed to a nice accident.

Solution to Example 1.5.



There are multiple ways of solving this problem, but let us try out new ideas. To employ the symmetry of the equilateral triangle, we rotate P about A by 60° to P'. Now, $AP = AP'$, and since B gets mapped to C, $BP = BC'$. Indeed, we can say $\triangle ABP \cong \triangle ACP'$.

Now, since $\angle PAP' = 60°$, PAP' is actually in equilateral triangle! So,

$$PP' = AP = 2BP = 2CP'.$$

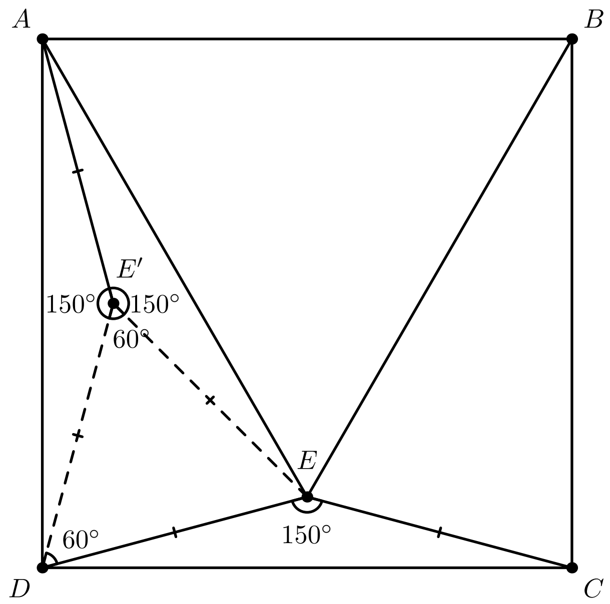
Furthermore,

$$\angle PP'C = \angle AP'C - \angle AP'P = 120° - 60° = 60°.$$

Recalling that $PP' = 2PC$, we get that $\triangle PP'C$ is actually a 30-60-90 triangle. Thus, $\angle CPP' = 30°$, and $\angle CPA = 90°$.

Solution to Example 1.6.

This is an example where everything feels nice. Clearly, E is well defined, and if we suppose $\triangle AEB$ is equilateral, all angels work nicely. However, that is all circular reasoning. Triangle angle sum alone gets us nowhere. $\angle DEA, \angle DAE$ remains a mystery.



Instead, let us try something smarter. We reflect E about BD to E'. Since C is mapped to A, $DE' = DE$ and $AE' = CE$.

Furthermore, $\angle ADE' = \angle EDC = 15^\circ$, which implies that $\angle EDE' = 60^\circ$, and $\triangle E'DE$ is equilateral. So, $EE' = DE'$!

Then, $\angle CE'E = 60^\circ$. Finally, as $\angle EE'A = 360^\circ - \angle AE'D - \angle CE'E = 150^\circ$.

Alas, we have another congruent angle formed! It follows that $\triangle AE'E \cong AE'D$, and $\angle E'AE = \angle DAE' = 15^\circ$, so $\angle EAB = 60^\circ$.

Similar reasoning shows that $\angle EBA = 60^\circ$, and $\triangle AEB$ is equilateral.

What was nice about this problem is that there turned out to be a hidden equilateral triangle among with congruent figures. This theme will show up repeatedly throughout the section.

Solution to Example 1.7.

Reflect N over M to N'. We have $\triangle N'MB \cong \triangle NMC$ by SAS. So, $\angle MBN' = \angle MCN = 90^\circ$, and N' is on AB.

Here, $\angle AMN' = \angle AMN = 90^\circ$, since $AM \perp MN$. Along with $MN' = MN$, we have $\triangle AMN' \cong AMN$ by SAS. Therefore, $\angle NAM = \angle N'AM$, and with B on AN', we have $\angle NAM = \angle N'AM = \angle BAM$, as desired.

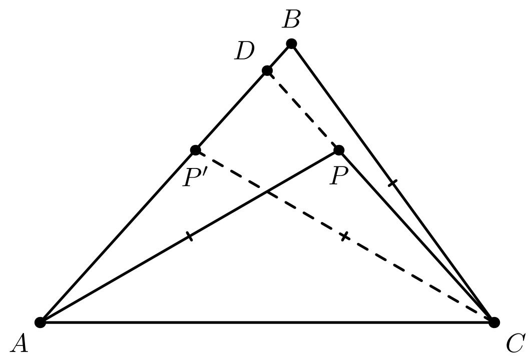
Solution to Example 1.8.

Extend PC to meet AB at D. $\angle BAC = \angle PCA = 48^\circ$, and $\triangle ADC$ is isosceles.

Reflect P over the line of symmetry of $\triangle ADC$ to P' (P' is on AD). Because $CP' = AP$ and $AP = BC$, we have $CP' = BC$, and $\triangle P'CB$ is isosceles.

That was the critical observation we needed to utilize congruence. Now, we finish the arithmetic:

$$\angle CP'D = \angle APD = \angle PAC + \angle PCA = 78^\circ.$$



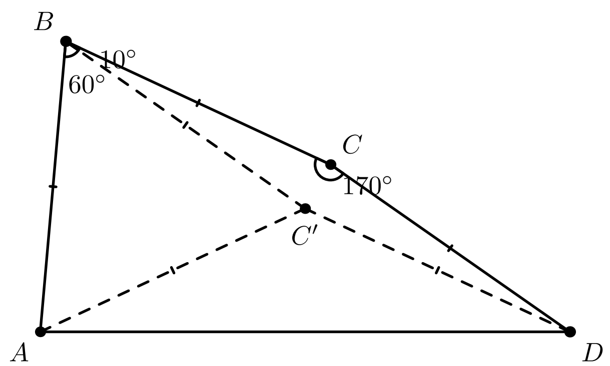
By using the fact that $\triangle P'CB$ is isosceles:

$$\angle P'CB = \frac{180^\circ - \angle CP'B}{2} = 24^\circ$$

Finally,

$$\angle PCB = \angle P'CB - \angle PCP' = 30^\circ - 24^\circ = 6^\circ.$$

Solution to Example 1.9.



Since $\angle CBD = 5^\circ$, and $\angle CBA = 70^\circ$, it seems like we can somehow make a 60° angle...

Reflect C over BD to C'. $BCDC'$ is a rhombus, and $\angle CBC' = 10^\circ$. Therefore, $C'BA = 60^\circ$. But $BC' = BC = AB$! So, $\triangle ABC'$ is equilateral, and $C'A = AB = CD = C'D$.

Since $\angle DC'A = 360^\circ - 170^\circ - 60^\circ = 130^\circ$, $C'AD = 25^\circ$. It follows that $\angle BAD = \angle C'AB + \angle C'AD = 60^\circ + 25^\circ = 85^\circ$.

Solution to Example 1.10.

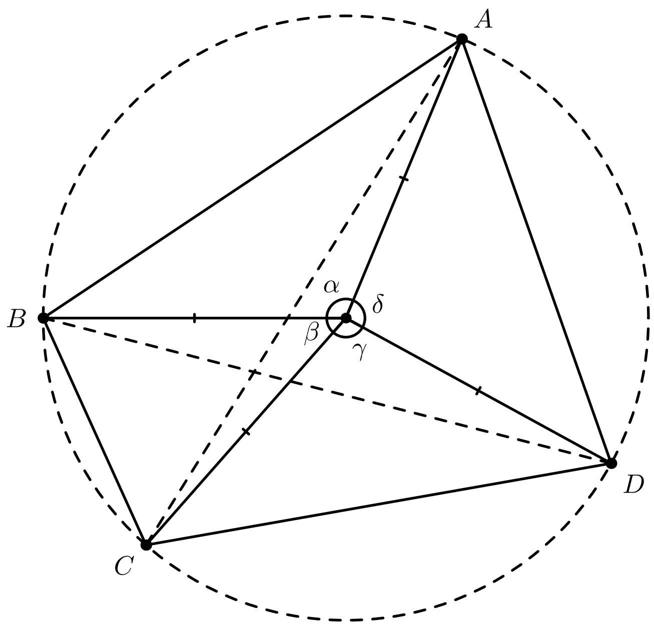
Since $OA = OB = OC = OD$, by definition $ABCD$ all lie on the same circle. Observe that we have 6 isosceles triangles in the diagram. Label

$$\angle AOB = \alpha, \angle BOC = \beta, \angle COD = \gamma, \angle DOA = \delta.$$

Note that $\alpha + \beta + \gamma + \delta = 360^\circ$.

We are interested in comparing $\angle ABD$ and $\angle ACD$. We write

$$\angle ABD = \angle ABO + \angle OBD = \frac{180^\circ - \alpha}{2} + \frac{180^\circ - \beta - \gamma}{2} = \frac{1}{2}(360^\circ - \alpha - \beta - \gamma) = \frac{1}{2}\delta$$



In a similar fashion,

$$\angle ACD = \angle ACO + \angle OCD = \frac{180^\circ - \alpha - \beta}{2} + \frac{180^\circ - \gamma}{2} + = \frac{1}{2}(360^\circ - \alpha - \beta - \gamma) = \frac{1}{2}\delta$$

Thus,

$$\angle ABD = \angle ACD.$$

Now, to compare $\angle ABC$ and $\angle ADC$, observe that

$$\angle ABC = \angle OBA + \angle OBC = \frac{180^\circ - \alpha}{2} + \frac{180^\circ - \beta}{2}$$

$$\angle ADC = \angle ODA + \angle ODC = \frac{180^\circ - \delta}{2} + \frac{180^\circ - \gamma}{2}$$

It follows that

$$\angle ABC + \angle ADC = \frac{720^\circ - \alpha - \beta - \gamma - \delta}{2} = 180^\circ.$$

So,

$$\angle ABC + \angle ADC = 180^\circ$$

What is really nice about these two results is that they are independent of O. Furthermore (though we won't prove this), any of these conditions are enough to establish that the four points A, B, C, D lie on the same circle!

So, we can actually say that

$$\angle ABC + \angle ADC = 180^\circ \iff \angle ABD = \angle ACD$$

assuming the configuration is correct (A, B, C, D being in that order).

Now, we are able to bypass isosceles triangles altogether!

Solution to Example 1.11.

Since $\angle ABD = \angle ACD$, the quadrilateral $ABCD$ is cyclic. That means that $\angle ACB = \angle ADB = 80^\circ$.

See how quick that solution was? In some ways, the congruent condition is a pure "coincidence" that forces everything to be nice.

If, say, $\angle ADB = 65^\circ$ but $\angle ACD = 70^\circ$, then we actually don't have a definitive solution. We can slide points around, and $\angle ACB$ change. Now, when $\angle ACD = 65^\circ$ again, points still can slide; however, $\angle ACD$ will remain fixed.

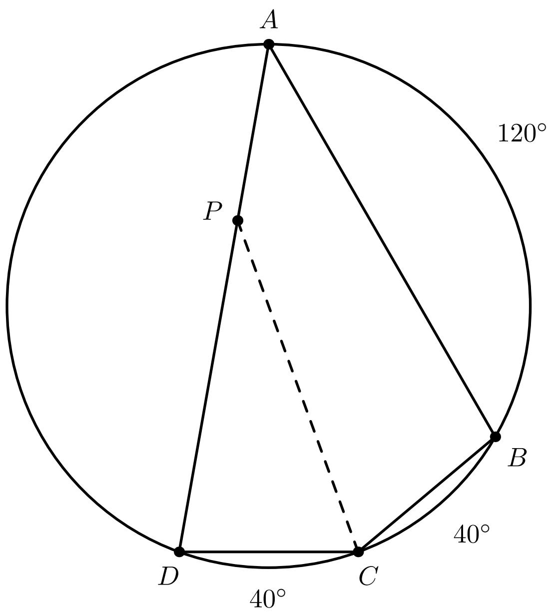
That is the power of the cyclic quadrilateral.

Solution to Example 1.12.

For this problem, it makes much more sense to work in arc measures. The first condition tells us that $\widehat{AC} = 2 \cdot \widehat{BD} = 160^\circ$. The second suggests that $\widehat{BC} = \widehat{CD}$. So,

$$\widehat{BC} + \widehat{CD} = \widehat{BD} = 80^\circ \implies \widehat{BC} = \widehat{CD} = 40^\circ.$$

Then, we can draw a diagram, and all relations become clear.



Since $ABCD$ is cyclic,

$$\angle BCD = 180^\circ - \angle DAB = 180^\circ - \frac{1}{2}\widehat{BD} = 140^\circ \implies \angle BCP = 70^\circ$$

Since P is on the angle bisector of $\angle BCD$, symmetry, we have that

$$\angle CBP = \angle CDP = \angle ADC = 80^\circ$$

Moreover,

$$\angle BPC = 180^\circ - \angle CBP - \angle BCP = 180^\circ - 80^\circ - 70^\circ = 30^\circ.$$

Our final answer is $\angle BPD = 2 \cdot BPC = 60^\circ$.

2 Solutions to Section 2 Examples

Solution to Example 2.1.

This example starts off out chapter with a little bit of algebraic manipulation.

Let D be on AB such that $ID \perp AB$. Since $ID \parallel EA \parallel FB$, we have $\angle BID = \angle BEA$ and $\angle BDI = \angle BAE$ by corresponding angles of parallel lines. So, by AA $\triangle BDI$ is similar to $\triangle BAE$. Likewise, $\triangle ADI \sim \triangle ABF$.

Now, we know that corresponding sides of similar triangles are in equal ratios. Our strategy is to use sides that either we need in our answer, sides that we already know, and sides that can be algebraically manipulated in an equation.

$$\frac{ID}{EA} = \frac{BD}{AB} \text{ and } \frac{ID}{FB} = \frac{AD}{AB}.$$

Here, we want to solve for ID, we know $EA = 20, FB = 30$. Finally, it appears that we can relate the two ratios on the right hand side of the equations.

$$\frac{BD}{AB} + \frac{AD}{AB} = \frac{AB}{AB} = 1 \implies \frac{ID}{20} + \frac{ID}{30} = 1$$

So, $ID = 12$. That was nice.

Solution to Example 2.2.

This time, the similar triangles are oriented not in the same direction, but oppositely. Since $AD \parallel BC$, we have that $\angle BPE = \angle PDA$ and $\angle PEB = \angle PAD$, by alternate interior angles. (We also have $\angle APD = \angle PEB$).

Thus, by AA similarity we have $\triangle BPE \sim \triangle DPA$, or

$$\frac{BP}{PD} = \frac{BE}{DA} = \frac{1}{3}.$$

A similar argument indicates $\triangle BQF \sim \triangle DQA$, and $BQ/DQ = 3/2$.

Now, let us write everything in a ratio to BD. We can use algebra to say

$$\frac{BP}{BD} = \frac{BP}{BP + DP} = \frac{1}{1 + \frac{DP}{BP}} = \frac{1}{1+3} = \frac{1}{4}.$$

However, no one really thinks this way. It is much direct to think as follows: Take $BP = k$. Then, $DP = 3k$, so $BD = BP + DP = 4k$, and $BP/BD = 1/4$.

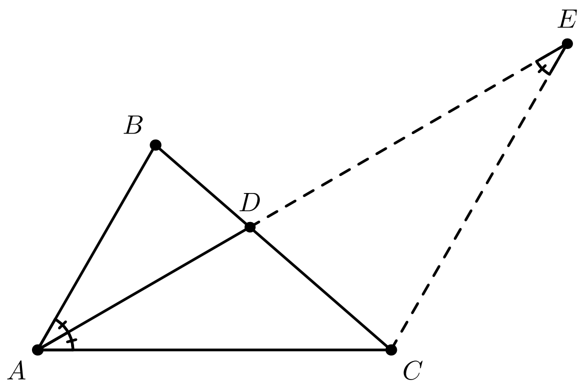
Nevertheless, we can also find $BQ/BD = 2/5$, and again we have $BP : PQ : QD = 5 : 3 : 12$.

Solution to Example 2.3.

Our intuition is to construct a parallel line somewhere, so let's see how it goes.

Construct line ℓ through C such that ℓ is parallel to AB, and let ℓ intersect AD at E. We label in the most useful congruent angles.

$$\angle CAD = \angle BAD = \angle CED$$



where the former comes from the angle-bisector condition and the later comes from parallel lines. We have an isosceles triangle!

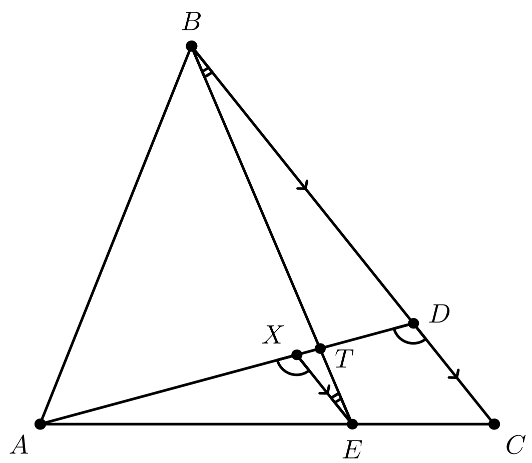
Now, using the most obvious pair of similar triangles formed by the parallel lines (there really is only one pair), $\triangle ABD \sim \triangle ECD$, and the fact that $AC = CE$, we get

$$\frac{AB}{BD} = \frac{EC}{CD} = \frac{AC}{CD}$$

and we are done!

Thus, we may say that in principle, the angle bisector theorem is just two similar triangles, but with one "shifted" to form an angle bisector.

Solution to Example 2.4.



It is not clear how BD, CD are related. We can, in a sense, say that BD and CD are 'parallel'. Once again, we need to introduce similar triangles via parallel lines. Let the line through E parallel to BC intersect AD at X.

Now, XE is parallel to both BD, CD, so we have two pairs of similar triangles (hence our construction is useful)

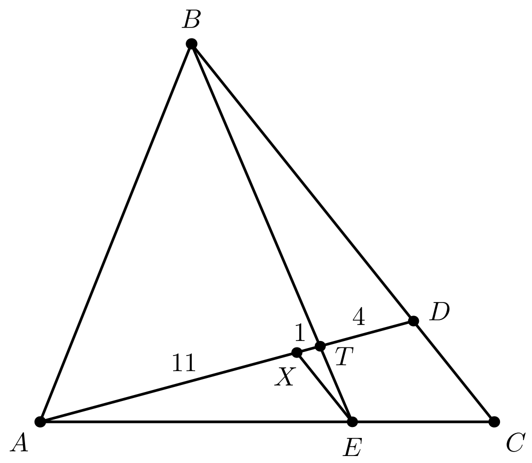
$$\triangle AXE \sim \triangle ADC \text{ and } \triangle XET \sim \triangle DBT$$

These similarities gives us

$$\frac{XE}{BD} = \frac{ET}{BT} \text{ and } \frac{XE}{CD} = \frac{AX}{AD}$$

We use both equations to cancel EX, and obtain

$$\frac{CD}{BD} = \frac{ET}{BT} \cdot \frac{AD}{AX}.$$



Now, it is given that $BT/ET = 4$. To find AD/AX, it is convenient to assign values. we can set $XT = 1$. Now, $TD = 4$, since

$$\triangle XET \sim DBT \implies \frac{TD}{XT} = \frac{BT}{ET} = 4$$

We are also given that $AT/TD = 3$, so it follows that $AT = 3 \cdot 4 = 12$, and $AX = 11$. Finally,

$$\frac{CD}{BD} = \frac{1}{4} \cdot \frac{16}{11} = \frac{4}{11}$$

That was quite long. However, this problem purposely picked in a sense the "worse" given ratios. Luckily, problems tend to give AE/EC or AE/AC, the later which is immediately equal to AT/AD, to simplify calculations.

We will also develop more theory regarding this configuration in Section 4.

Solution to Example 2.5.

Now, we are given $MB = MC$, but there is no apparent way we can use this condition directly. Furthermore, $\angle ANM = 90^\circ$ doesn't line up with any sides of a right triangle.

Therefore, let us extend DM to meet AB at E. By AAS we have that $\triangle BME \cong \triangle CMD$, which implies that $BE = DC$ (here, $BECD$ is a parallelogram)

Also, $\triangle ANE$ is a right triangle. However, recall that for a right triangle, the distance from the right angle vertex to the midpoint of the hypotenuse is equal to the half the hypotenuse length (consequence of Thales's theorem)

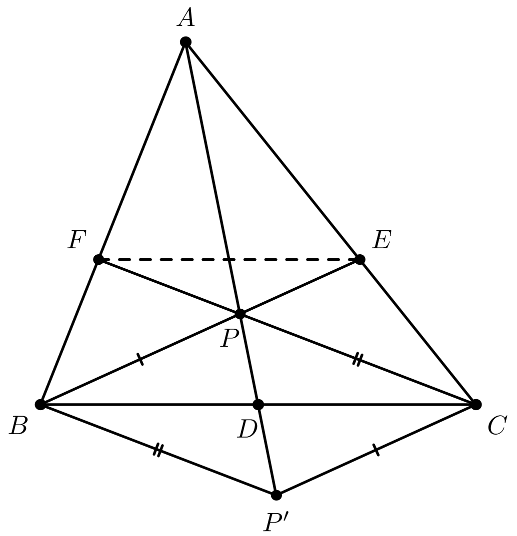
But B is the midpoint of $\triangle ANE$! So, it follows that

$$NB = BE = BA.$$

Solution to Example 2.6.

This is another problem where everything should be nice, but we lack a way to force everything to be equal. Here, the parallelogram trick is the way.

Take P' on the extension of AD such that $DP' = DP$. Then, since $BD = DC$, we have that $BPCP'$ is a parallelogram.



Now, since $BPCP'$ is a parallelogram, we know that $BP' \parallel FP$ and $CP' \parallel EP$, and so

$$\triangle AFP \sim \triangle ABP' \text{ and } \triangle AEP \sim \triangle ACP'$$

Looking at ratios, we have

$$\frac{FP}{BP'} = \frac{AP}{AP} = \frac{EP}{CP'}$$

But $BP' = CP$ and $CP' = BP$. Then,

$$\frac{FP}{CP} = \frac{EP}{BP} \implies \triangle EFP \sim \triangle BCP.$$

It follows that $EF \parallel BC$. That was neat. This is one of the few times we actually cleverly manipulate lengths to prove angle conditions!

As a slight foreshadow, we will see this configuration again in **Problem 2.8**, and more generally as part of **Example 4.6**.

Solution to Example 2.7.

Recall that in a cyclic quadrilateral, angles inscribing the same arc are congruent, and opposite angles are supplementary.

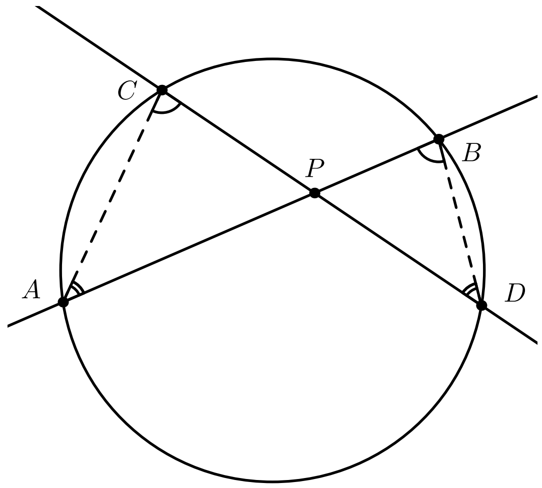
Let us first consider when P is inside ω. Observe that

$$\angle ACD = \angle ABD = \frac{1}{2}\widehat{AD}$$

Therefore, $\angle ACP = \angle DBP$. (we just changed the name of the angle, since P lies on both lines)

By similar reasoning,

$$\angle CAB = \angle BDC = \frac{1}{2}\widehat{CB}$$



Here, $\angle CAP = \angle BDP$.

Thus, by AA similarity,

$$\triangle ACP \sim DBP$$

This is the heart of Power of a Point. Essentially, the five points of intersection of two lines and a circle gives a pair similar triangles.

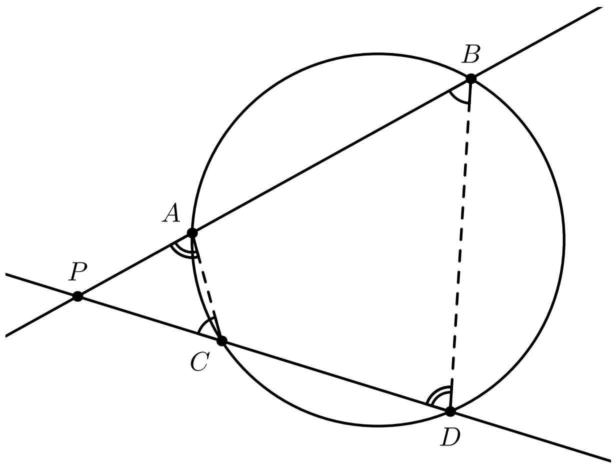
It follows that

$$\frac{CP}{AP} = \frac{BP}{DP} \implies CP \cdot DP = AP \cdot BP$$

Observe that if we instead connected AD, CB, then we will find

$$\triangle ADP \sim CBP$$

which is also true! This formulation can also give the same Power of a Point expression.



Now, we turn our attention to the case that P is outside ω. This time,

$$\angle ACD + \angle ABD = 180^\circ \quad \text{and} \quad \angle CAB + \angle CDB = 180^\circ$$

where opposite angles are supplementary. But $\angle PAC$ and $\angle PCA$ are supplementary to $\angle CAB$ and $\angle ACD$, respectively. So,

$$\angle PCA = \angle PBD \quad \text{and} \quad \angle PAC = \angle PDB.$$

Again, with AA similarity, we have

$$\triangle PCA \sim \triangle PBD \implies PA \cdot PB = PC \cdot PD.$$

It is also true that $\triangle PAD \sim \triangle PCB$, and our Power of a Point statement is also the same.

Since $PA \cdot PB$ is constant for any chosen line m, we can write

$$PA \cdot PB = Power_\omega(P),$$

where $Power_\omega(P)$ is only a function of P and ω.

Finally, when line m is a diameter, and P is inside the circle, we write

$$Power_\omega(P) = PA \cdot PB = (R - OP) \cdot (R + OP) = R^2 - OP^2$$

We are therefore able to extract the distance OP. Thus, if we ever needed to find OP, we can instead use R, and $PC \cdot PD$ for any other (convenient) line.

Solution to Example 2.8.

First, we need to make the observation that P is the circumcenter of $\triangle ABC$.

It is easier to see this by first fixing isosceles $\triangle APB$. Then, there is a circle with center P that passes through A, B. $\angle APB$ is a center angle with $\angle APB = \widehat{AB}$. As $\angle ACB = \frac{1}{2}\widehat{AB}$, it forces C to be on the same circle.

To set up Power of a Point, extend BD to meet $\odot ABC$ (the circumcircle of ABC) at E. BE is a diameter, so $BE = 6$ and $DE = 5$.

Now, we use power of a point:

$$AD \cdot CD = ED \cdot BD = 5 \cdot 1 = 5$$

Make sure that you always set Power of a Point with 4 points on the circle! Sometimes people have the tendency to forget, if the fourth point isn't drawn in.

Solution to Example 2.9.

Once again (notice that this is becoming a theme of this section), we want to construct two pairs of similar triangles while only drawing one new point/line. Luckily, the numerous amount of given congruent angles allows for this to happen.

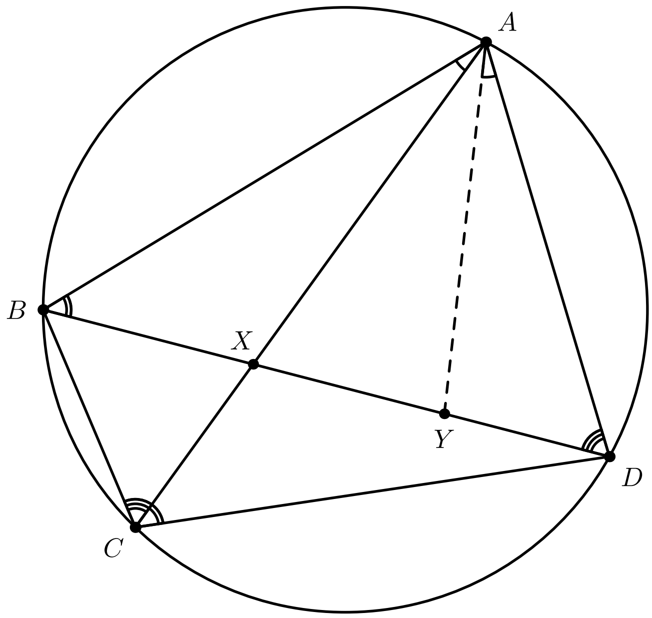
Let X be the intersection of the diagonals. Let us take Y on BD with $\angle BAC = \angle DAY$. Note that this also implies $\angle BAY = \angle DAX$.

Recall that $\angle BCA = \angle YDA$ and $\angle ABY = \angle ACD$. Therefore,

$$\triangle ABC \sim \triangle AYD \text{ and } \triangle ABY \sim \triangle ACD$$

This looks good enough. Now, we choose two clever pairs of side lengths to extract the similarity. BY and YD is a good choice, since $BY + YD = BD$, a desired quantity. Also, we don't want AY, since we don't know any way it relates to something else.

$$\frac{BC}{AC} = \frac{YD}{AD} \text{ and } \frac{BY}{AB} = \frac{CD}{AD}$$



Adding the two equations together

$$BD = BY + YD = \frac{AD \cdot BC + AB \cdot DC}{AC} \quad \text{or} \quad AC \cdot BD = AD \cdot BC + AB \cdot DC$$

The flipping the angle construction was nice, which allowed convenient similar triangles to be formed.

Solution to Example 2.10.

We set up Ptolemey's Theorem:

$$AB \cdot CD + AC \cdot BD = AD \cdot BC$$

Note that since D lies on the angle bisector of BAC, D must be the midpoint of $\widehat{BC}$. So, $BD = CD$. Rearranging the expression and plugging in values give

$$7 \cdot CD + 8 \cdot CD = AD \cdot 9 \implies \frac{AD}{CD} = \frac{5}{3}$$

Solution to Example 2.11.

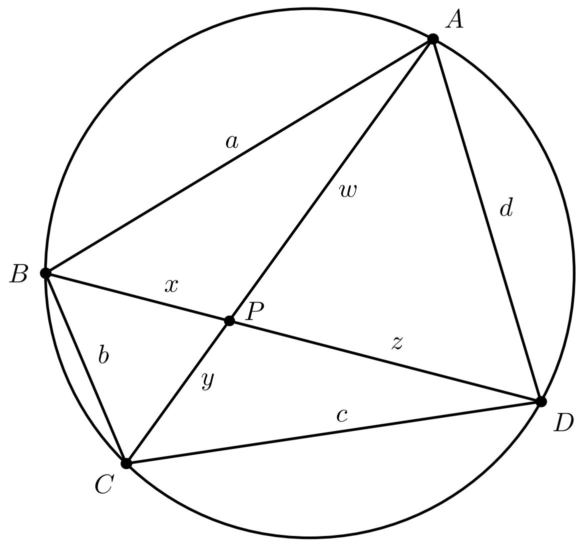
This time, we need to use similar triangle to greater depth, and utilize some clever algebra tricks to make things nice.

$$w = x\frac{d}{b} \quad \text{and} \quad y = x\frac{c}{a}$$

$$w = z\frac{a}{c} \quad \text{and} \quad y = z\frac{b}{d}$$

We want a linear relation between $w + y$ and $x + z$, so lets first write x and z individually in terms of $w + y$.

$$x = \frac{w + y}{\frac{d}{b} + \frac{c}{a}} \quad \text{and} \quad z = \frac{w + y}{\frac{a}{c} + \frac{b}{d}}$$



Thus, we can combine the two equations to get

$$\begin{aligned}\frac{BD}{AC} &= \frac{x+z}{w+y} \\ &= \frac{1}{\frac{a}{c}+\frac{b}{d}} + \frac{1}{\frac{d}{b}+\frac{c}{a}} \\ &= \frac{cd}{bc+ad} + \frac{ab}{bc+ad} \\ &= \frac{ab+cd}{ad+bc}\end{aligned}$$

Now, we can instead substitute this back into Ptolemey's.

$$AC^2 = \frac{(ac+bd)\cdot(ad+bc)}{(ab+cd)}$$

Now, while I do think that this is one of the very few longer results worth remembering, it is quite confusing to remember which goes where. We may remember the bottom term as the term with two side length pairs on the same side of AC.

Solution to Example 2.12.

It is given that X, O, Y, Z are cyclic (on circle C_2). Furthermore, we must have $XO = OY = r$. Then, the most straightforward approach is to use the stronger form of Ptolemy's theorem.

$$\begin{aligned}OZ^2 &= \frac{(XZ\cdot OY + XO\cdot YZ)\,(XZ\cdot ZY + XO\cdot OY)}{XZ\cdot XO + YZ\cdot YO} \\ 11^2 &= \frac{20r\cdot\left(r^2+13\cdot 7\right)}{20r}\end{aligned}$$

So, it turns out that $r = \sqrt{30}$.

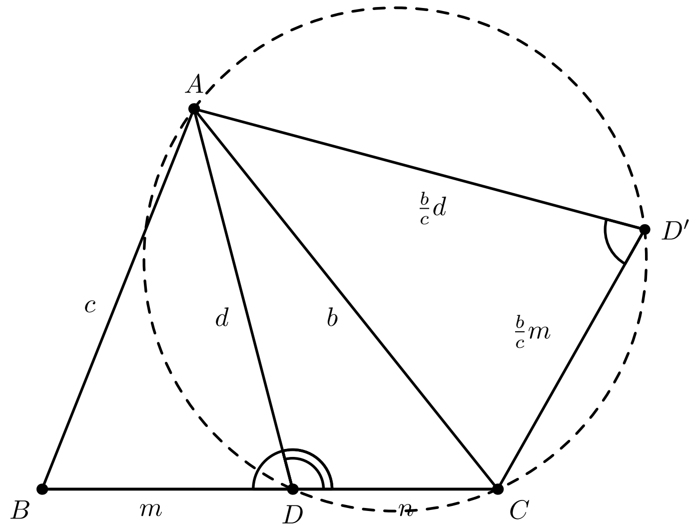
Solution to Example 2.13.

Look at $\triangle ADB$ and $\triangle ADC$ separately. What special relations exist between the two triangles?

One might naively say $AD = AD$, obviously. However, this really isn't important at all, since we have the power of similarity. We can scale things all we want. What really is special is that $\angle ADB + \angle ADC = 180°$. This is necessary for the two triangles to form a bigger triangle.

Doesn't supplementary angles remind us of something? Yes, we can actually form a cyclic quadrilateral!

Let us rotate $\triangle ADB$ about point A through an angle of $\angle CAD$, and also scaled by AC/AB. Here, B' overlaps with C, and we have a new D'.



Now, since $\angle AD'B' = \angle ADB$, we have a cyclic quadrilateral $AD'CD$, with lengths

$$AD' = \frac{b}{c}d, D'C = \frac{b}{c}m, CD = n, DA = d, AC = b.$$

Now, we finish with the stronger form of Ptolemey's.

$$b^2 = \frac{\left(\frac{b}{c}dn + \frac{b}{c}md\right) \cdot \left(\frac{b}{c}d^2 + \frac{b}{c}mn\right)}{\left(\left(\frac{b}{c}\right)^2 md + dn\right)}$$

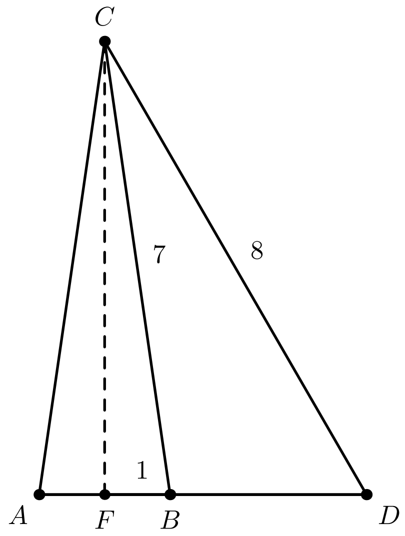
$$\left(\left(\frac{b}{c}\right)^2 md + dn\right) \cdot b^2 = \left(\frac{b}{c}\right)^2 \cdot d \cdot (m+n) \cdot (d^2 + mn)$$

$$b^2m + c^2n = d^2a + amn$$

This result is frequently used in computational problems.

3 Solutions to Section 3 Examples

Solution to Example 3.1.



Unlike proportionality, using the Pythagorean Theorem requires enough information. Typically, the problem should be solvable if it feels well defined, and enough lengths are given.

Here, we know enough lengths to start with, so the rest should just come naturally. Let the foot of altitude from C to AB be F. By the Pythagorean Theorem, we have

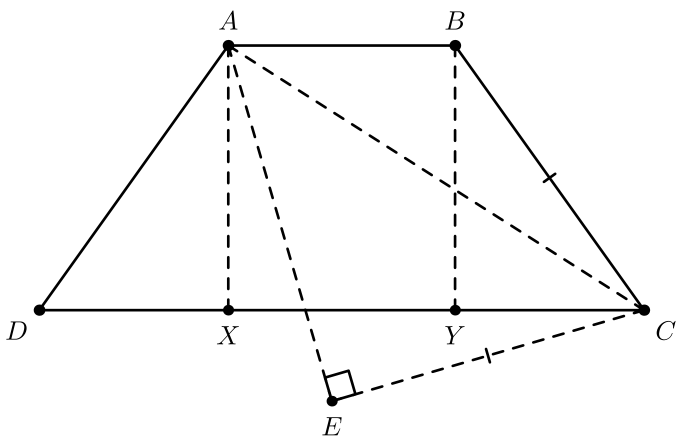
$$CF^2 + BF^2 = CD^2 \quad \text{and} \quad CF^2 + FD^2 = CD^2$$

We can solve for CF, though we don't need to. Subtract the two equations, and we have

$$\begin{aligned} DF^2 - BF^2 &= CD^2 - CB^2 \\ DF^2 &= 8^2 - 7^2 + 1^2 \\ DF &= 4 \end{aligned}$$

It follows that $BD = DF - BF = 3$.

Solution to Example 3.2.



Drop foots of altitude from A, B to X, Y on CD, respectively. Since $AE \perp EC$, we have

$$AE^2 = AC^2 - EC^2$$

Since we are given that $EC = BC$, we have

$$AE^2 = AC^2 - BC^2$$

This is nice. Our answer no longer depends on E. We might be worried that we can't find AC or BC, because we lack any information about the height of the trapezoid. However, it turns out that we don't need the height, since it cancels out.

$$AC^2 = AX^2 + CX^2 \quad \text{and} \quad BC^2 = BX^2 + CY^2$$

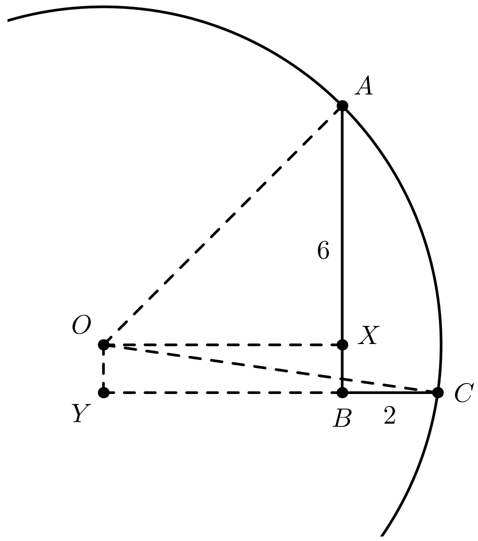
Here, $AX = BX$, and $CX = 11/2, CY = 5/2$. Then,

$$AE^2 = AX^2 + CX^2 - BY^2 - CY^2 = CX^2 - CY^2 = (11/2)^2 - (5/2)^2 = 24.$$

Our answer is $AE = 2\sqrt{6}$.

This example is typical of a relatively straightforward problem. The first part is the setup, and use the given conditions of the problem. After that, we get our answer in terms of some lengths that are easily computable. We finish with computation.

Solution to Example 3.3.



Let X, Y be the foots of altitude from O to AB, BC, respectively. $OXBY$ is a rectangle. Denote $OX = x, OY = y$.

By the construction of a circle, $OA = OC = \sqrt{50}$. Also, $\triangle OXA$ and $\triangle OYC$ are right triangles.

$$x^2 + (6 - y)^2 = 50$$
$$(x + 2)^2 + y^2 = 50$$

This is enough to solve for x, y individually. We find that $x = 5, y = 1$, and $OB = \sqrt{x^2 + y^2} = \sqrt{26}$.

Solution to Example 3.4.

This problem involves a similar radius setep, though we have to be careful about configuration issues: whether the two chords are on the same side of the circle or not.

Let us suppose that the distances from the chord to the center are x, y.

$$x = \sqrt{r^2 - 5^2} \quad \text{and} \quad y = \sqrt{r^2 - 7^2}$$

We know that either $x + y = 6$ or $x - y = 6$. We add the two equations to get

$$\sqrt{r^2 - 5^2} \pm \sqrt{r^2 - 7^2} = 6$$

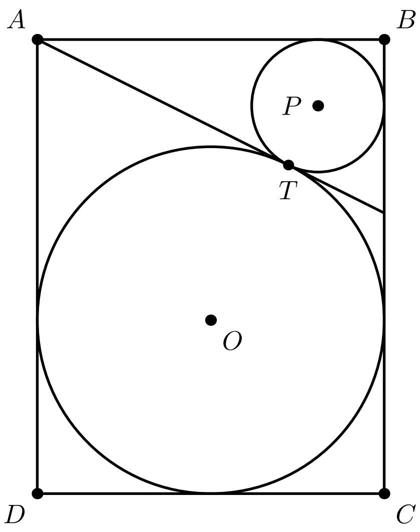
where $+$ corresponds to opposite of center, and $-$ corresponds to same side of center.

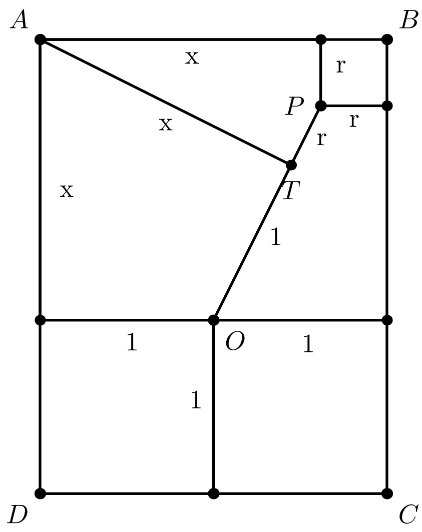
$$\begin{aligned}\left(\sqrt{r^2 - 5^2}\right)^2 &= \left(6 \mp \sqrt{r^2 - 7^2}\right)^2 \\ r^2 - 5^2 &= 36 \mp 12\sqrt{r^2 - 14^2} + r^2 - 7^2 \\ 0 &= 12 \mp 12\sqrt{r^2 - 7^2}\end{aligned}$$

The r^2 terms conveniently cancel (as usual). We end up needing $-$ of $\mp$, which means that it is $+$ initially, corresponding to the two chords being opposite of center. Also, $r = \sqrt{50}$, $x = 5$, $y = 1$.

We are interested in a chord in-between the two, so the distance from the chord to the center must be 2. It follows that the chord length is $2\sqrt{50 - 4} = 2\sqrt{46}$.

Solution to Example 3.5.





The key to this problem is simply to label all known lengths and relations. One key observation is that O, T, P are collinear. Furthermore, we use the tangent segments are congruent property to deduce that the three tangents from A all have the same length x.

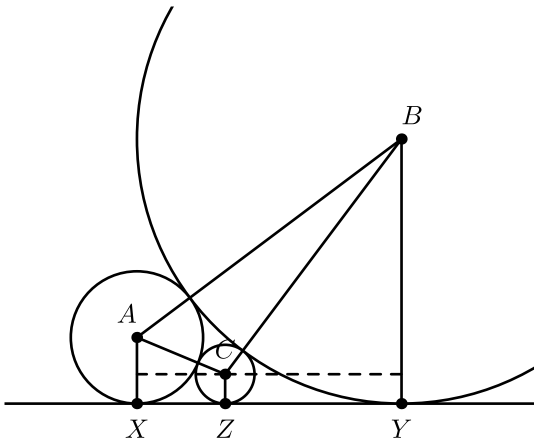
Now, consider the right triangle with hypotenuse OP and legs parallel to $ABCD$. We deduce that

$$(1 - r)^2 + (x - r)^2 = (1 + r)^2$$

Looking at AB, we have $x + r = 2$. Thus, we may substitute and solve for r, in which

$$r = \frac{3 \pm \sqrt{5}}{2} \quad \Longrightarrow \quad r = \frac{3 - \sqrt{5}}{2}.$$

Solution to Example 3.6.



Let the unknown radius be r. Drop perpendiculars from A, B, C down to X, Y, Z, respectively. By the Pythagorean theorem on the three right triangles with hypotenuse being the line that joins two triangle centers,

$$XY^2 + (4-1)^2 = (4+1)^2$$
$$XZ^2 + (1-r)^2 = (1+r)^2$$
$$ZY^2 + (4-r)^4 = (4+r)^2$$

It is straightforward to solve individually (as square terms cancel) $XY = 4, XZ = \sqrt{4r}, ZY = \sqrt{16r}$. Since $XY = XZ + ZY$, we find that $r = \frac{4}{9}$.

Solution to Example 3.7.

Without loss of generality, let $AD = 10, BC = 14$. Drop altitudes from C, D down to line AB, and let the foot of altitudes be E, F, respectively.

Setting the height of the trapezoid to be h, we have $CE = DF = h$. So,

$$AF = \sqrt{AD^2 - DF^2} = \sqrt{10^2 - h^2} \text{ and } EB = \sqrt{CB^2 - CE^2} = \sqrt{14^2 - h^2}$$

To relate the two, we use $AF + FE + EB = 33$, or

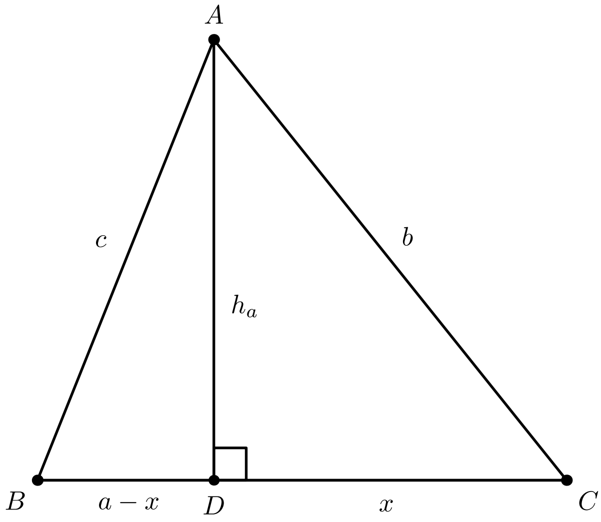
$$\sqrt{10^2 - h^2} + 21 + \sqrt{14^2 - h^2} = 33.$$

Once again, this actually isn't too bad to solve, since the square terms cancel.

$$14^2 - h^2 = 12^2 - 24\sqrt{10^2 - h^2} + 10^2 - h^2 \implies h = 4\sqrt{6}$$

Plugging h back in, we have $AF = 2, EB = 10$. To form a diagonal, we can take the altitude h, plus base AE or FB. Here, the shorter diagonal is desired, which is

$$AC = \sqrt{AE^2 + CE^2} = \sqrt{23^2 + \left(4\sqrt{6}\right)^2} = 25.$$



Solution to Example 3.8.

Just some algebra.

$$\begin{aligned} b^2 - x^2 &= c^2 - (a-x)^2 = h_a^2 \\ b^2 &= c^2 - a^2 + 2ax \\ \frac{b^2 + a^2 - c^2}{2a} &= x \end{aligned}$$

We continue to find h_a.

$$\begin{aligned} h_a^2 &= b^2 - x^2 \\ &= \frac{4a^2b^2 - \left(b^2 + a^2 - c^2\right)^2}{4a^2} \\ &= \frac{\left(2ab + b^2 + a^2 - c^2\right)\left(2ab - b^2 - a^2 + c^2\right)}{4a^2} \\ &= \frac{\left((a+b)^2 - c^2\right)\left(-(a-b)^2 + c^2\right)}{4a^2} \\ &= \frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4a^2} \end{aligned}$$

and we are done! We used differences of squares to factorize the expression multiple times.

Now, you might have been bothered with the $4a^2$ on the bottom. Wouldn't everything be nicer if we multiplied it to the other side of the equation? Indeed, we will see what happens soon.

Solution to Example 3.9.

By Heron's formula, we have

$$4 \cdot 8 \cdot AH = \sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)} = \sqrt{24 \cdot 10 \cdot 6 \cdot 8}$$

So, $AH = 3\sqrt{5}$, and $BH = 2$, $HC = 6$.

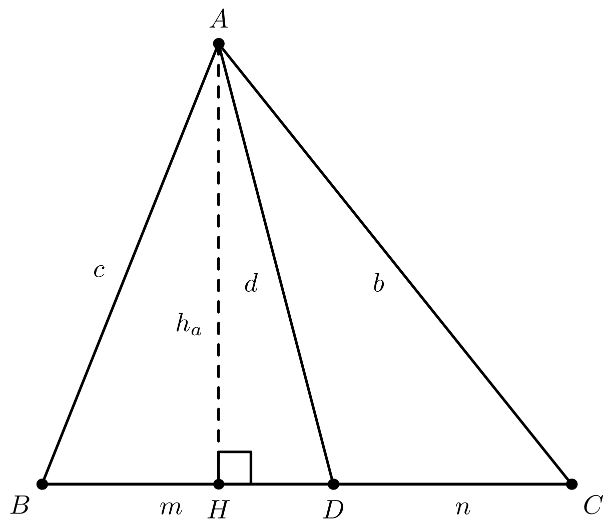
Using the angle bisector theorem on $\triangle ABH$ and $\triangle ACH$, we find

$$\frac{AP}{PH} = \frac{7}{2} \implies \frac{AP}{AH} = \frac{7}{9} \text{ and } \frac{AQ}{QH} = \frac{9}{6} \implies \frac{AQ}{AH} = \frac{3}{5}$$

Finally, we obtain

$$QP = AH \cdot \left(\frac{AP}{AH} - \frac{AQ}{AH}\right) = 3\sqrt{5}\left(\frac{7}{9} - \frac{3}{5}\right) = \frac{8\sqrt{5}}{15}.$$

Solution to Example 3.10.



All we need to do is write the critical Pythagorean Theorem.

$$AD^2 = HA^2 + HD^2 = AC^2 - HC^2 + HD^2$$

Now, recall from **Example 3.8** that

$$HC = \frac{b^2 + a^2 - c^2}{2a}$$

The rest of the algebraic manipulation is straightforward. (Note: We didn't write out the full Heron's formula, but instead left $h_a^2 = b^2 - x^2$ to simplify the algebra.)

$$\begin{aligned}
AT^2 &= HA^2 + HD^2 = AC^2 - HC^2 + HD^2 \\
d^2 &= b^2 - \left(\frac{b^2 + a^2 - c^2}{2a}\right)^2 + \left(\frac{b^2 + a^2 - c^2}{2a} - n\right)^2 \\
&= b^2 + n^2 - 2n\frac{b^2 + a^2 - c^2}{2a} \\
d^2a &= b^2a + n^2a - n(b^2 + a^2 - c^2) \\
d^2a &= n^2c + b^2(a - n) - an(a - n) \\
d^2a + anm &= b^2m + c^2n
\end{aligned}$$

This agrees with what we found in **Example 2.13**

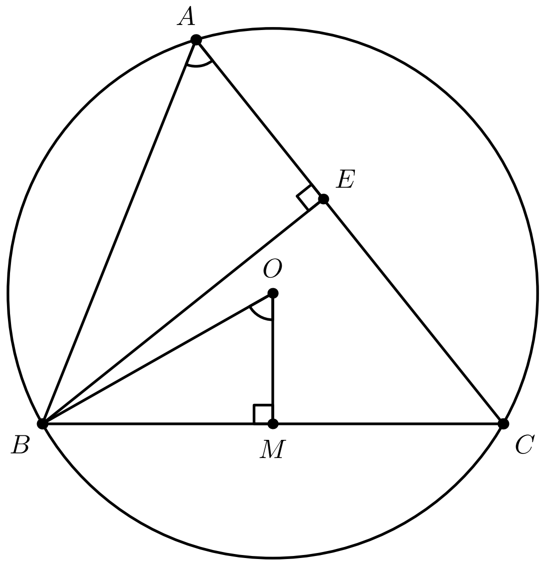
Solution to Example 3.11.

Let D, E be the foot of altitudes from A, B, to BC, AC, respectively. Obviously, $\angle ADC = \angle BEC = 90^\circ$, and $\angle DCA = \angle ECB = \angle C$. Therefore, $\triangle ADC \sim \triangle BEC$.

Then,

$$\frac{AD}{AC} = \frac{BE}{BC} \implies AD \cdot BC = BE \cdot AC \implies a \cdot h_a = b \cdot h_b$$

Solution to Example 3.12.



Let E be the foot from B to AC, and M be the midpoint of BC. By properties of inscribed angles in a circle, we have $\angle BAC = \frac{1}{2}\angle BOC$. But $\angle BOM = \frac{1}{2}\angle BOC$, so $\angle BAC = \angle BOM$.

Also, $\angle OMB = \angle AEB = 90°$, so $\triangle OMB \sim \triangle AEB$. Thus,

$$\frac{OB}{BM} = \frac{AB}{BE} \implies R = OB = \frac{AB}{BE} \cdot \frac{BC}{2} = \frac{c}{(2A/b)} \cdot \frac{a}{2} = \frac{abc}{4A}$$

Solution to Example 3.13.

Have the incircle touch BC, CA, AB at X, Y, Z, respectively. We have derived the lengths of segments AY, YC, CX, XB, BZ, ZA in problem 1.7. However, we may re-derive it with the equal tangents property.

Since AY, AZ are tangents from A to the incircle, $AY = AZ$. Define $AY = AZ = \ell_a$, and similarly define ℓ_b, ℓ_c.

$$\ell_a + \ell_b = c$$
$$\ell_b + \ell_c = a$$
$$\ell_c + \ell_a = b$$

Solving the system yields

$$\ell_a = \frac{-a+b+c}{2}, \ell_b = \frac{a-cb+c}{2}, \ell_c = \frac{a+b-c}{2},$$

which is handy to know.

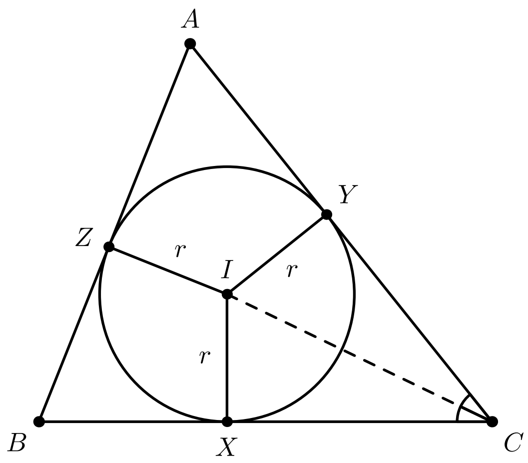
Now, since $\angle ICX = \angle C/2$, we are motivated to draw similar triangles. Let AD be an altitude, and have AD intersect IC at P.

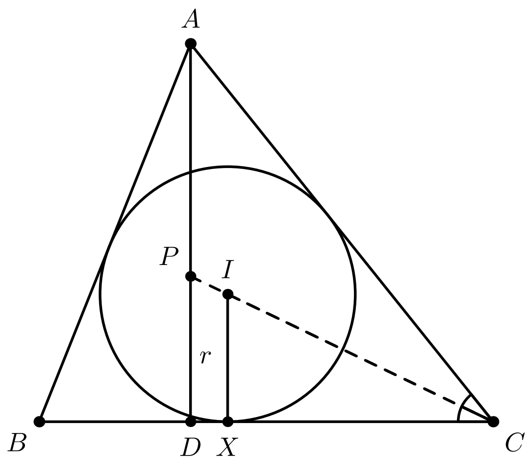
Since IC is an angle bisector, we use the angle bisector theorem,

$$\frac{PD}{PA} = \frac{CD}{CA} \implies \frac{PD}{AD} = \frac{CD}{CD+AC}$$

Now, we express PD both using similar triangles to IX and with the angle bisector theorem.

$$AD \cdot \frac{CD}{CD+AC} = PD = r \cdot \frac{CD}{XC}$$





We equate the two sides and obtain

$$\begin{aligned}
r \cdot \frac{CD}{XC} &= AD \cdot \frac{CD}{CD+AC} \\
r \cdot (CD + AC) &= AD \cdot XC \\
r \cdot \left(b + \frac{a^2+b^2-c^2}{2a}\right) &= h_a \cdot \left(\frac{a+b-c}{2}\right) \\
r \cdot \frac{(a+b+c)(a+b-c)}{2a} &= h_a \cdot \left(\frac{a+b-c}{2}\right) \\
r(a+b+c) &= ah_a \\
rs &= A
\end{aligned}$$

where a, b, c terms cancel nicely, and we are left only with the semiperimeter and area.

Now, if we are allowed to use the additive property of area, we can consider the triangle areas $[AIB], [BIC], [CIB]$. Observe that $\triangle BIC$ has altitude r and base a, so $[AIB] = ra/2$. Similar relations hold for the other two triangles. Then, adding the areas give

$$A = [ABC] = [AIB] + [BIC] + [CIA] = \frac{rc}{2} + \frac{ra}{2} + \frac{rb}{2} = rs$$

Solution to Example 3.14.

With our tools, this problem becomes very straightforward.

$$IN = r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \sqrt{\frac{12 \cdot 13 \cdot 14}{39}} = 2\sqrt{14}.$$

Also, $BN = (25 + 27 - 26)/2 = 13$, so

$$BI = \sqrt{\left(2\sqrt{14}\right)^2 + 13^2} = 15.$$

Solution to Example 3.15. Recall that for a rhombus, $AC \perp BD$. Furthermore,

$$[ABC] = [ABD] = (AC \cdot BD)/4.$$

We have

$$R_{\triangle ABD} = \frac{AD \cdot AB \cdot BD}{4[ABD]} \quad \text{and} \quad R_{\triangle ABC} = \frac{AB \cdot BC \cdot AC}{4[ABC]}$$

$$\frac{12.5}{25} = \frac{R_{\triangle ABD}}{R_{\triangle ABC}} = \frac{BD}{AC}$$

Letting $BD = x$, we get $AC = 2x$ and the side length of the rhombus

$$AB = BC = CD = DA = \sqrt{\left(\frac{x}{2}\right)^2 + x^2} = \frac{\sqrt{5}}{2}x.$$

Therefore, we can go back and solve for x.

$$12.5 = R_{\triangle ABD} = \frac{\left(\frac{\sqrt{5}}{2}x\right) \cdot \left(\frac{\sqrt{5}}{2}x\right) \cdot x}{x \cdot 2x} \implies x = 20$$

Our final answer is

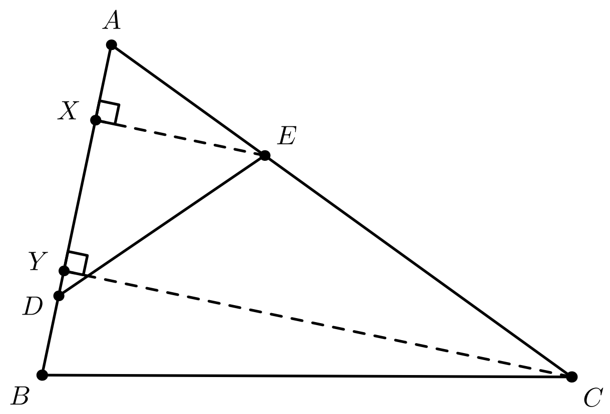
$$[ABCD] = 2[ABD] = 2 \cdot \frac{x \cdot 2x}{4} = 400.$$

4 Solutions to Section 4 Examples

Solution to Example 4.1.

It is perhaps reasonable to try to compute the are ratio $[ADE]/[ABC]$ first, since we are comfortable with triangle areas.

Let us drop altitudes from E, C to AB, and call the foots X, Y, respectively. The nice thing is that we have similar right triangles - $\triangle AXE \sim \triangle AYC$ which is enough to solve the problem.



$$\frac{[ADE]}{[ABC]} = \frac{AD \cdot EX}{AB \cdot CY} = \frac{AD}{AB} \cdot \frac{AE}{AC} = \frac{19}{75}$$

This is nice and generalizable. We basically found that for two triangles sharing an angle, the ratio of the areas is simply the ratio of the product of adjacent side lengths.

Our answer to the problem is

$$\frac{[BCED]}{[ABC]} = 1 - \frac{[ADE]}{[ABC]} = \frac{56}{75} \implies \frac{[ADE]}{[BCED]} = \frac{16}{56}.$$

Solution to Example 4.2.

For this example, we will need our result from the previous example, along with the fact that area is additive.

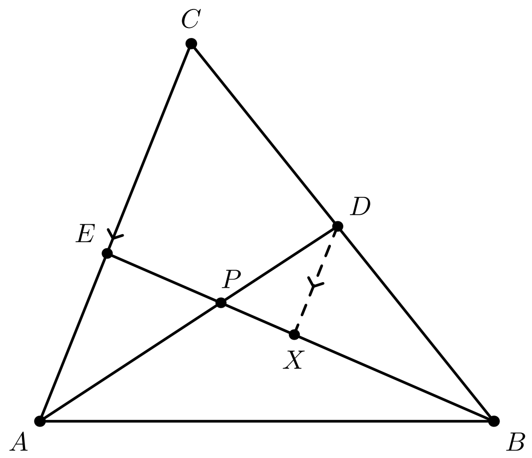
$$\frac{[BDF] + [FAB] + [BCD] + [DEF]}{[ACE]} = 1$$

The ratios for the other three triangles and the larger $\triangle ACE$ can be found easily. In fact, they are all the same.

$$\frac{[FAB]}{[ACE]} = \frac{FA \cdot AB}{AC \cdot AE} = \frac{3}{16}, \text{ likewise } \frac{[BCD]}{[ACE]} = \frac{3}{16} \text{ and } \frac{[DEF]}{[ACE]} = \frac{3}{16}$$

So, we get

$$\frac{[BDF]}{[ACE]} + 3 \cdot \frac{3}{16} = 1 \implies \frac{[BDF]}{[ACE]} = \frac{7}{16}.$$



Solution to Example 4.3.

We use the same trick pioneered in **Example 2.4**. Construct line ℓ through D parallel to AC, and let ℓ intersect BE at X. Due to the parallel condition,

$$\triangle AEP \sim \triangle DXP \quad \text{and} \quad \triangle BXD \sim \triangle BEC.$$

Our strategy is to compare XD to AE and EC.

$$\frac{XD}{AE} = \frac{DP}{AP} \quad \text{and} \quad \frac{XD}{EC} = \frac{BD}{BC}.$$

Eliminating the XD term gives

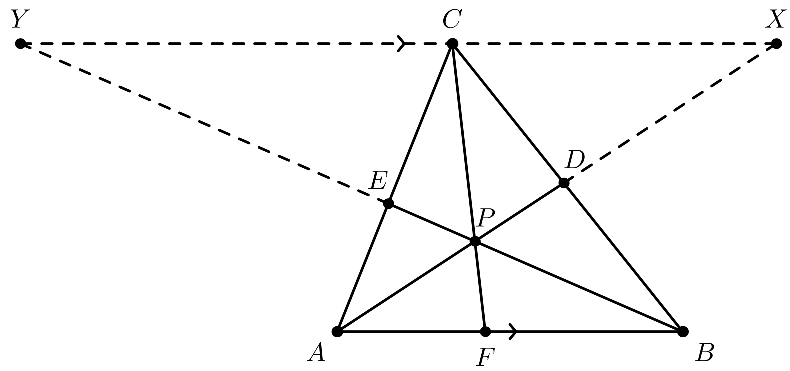
$$\frac{AE}{EC} = \frac{AP}{DP}\frac{BD}{BC} \implies \frac{AP}{AD} \cdot \frac{BD}{BC} \cdot \frac{EC}{CA} = 1,$$

as desired.

This formula is admittedly a bit long, and not that straightforward to use. I personally recommend sticking with the parallel line trick for numerical problems, especially since the parallel line trick can adapt to other possible inputs (such as the conditions in **Example 2.4**).

However, this theorem's converse is also true, so it can also be used to prove that the three cevians are collinear. Finally, many text gives the equation with a -1, to indicate directed lengths. As stated in the introduction, we won't bother with configurations here.

Solution to Example 4.4.



Once again, we will use the parallel line trick. Hopefully it starts to becomes obvious where to draw the line. Take line ℓ through C parallel to AB, and let the extensions of AD, BE meet ℓ at X, Y, respectively. Now,

$$\triangle ABD \sim \triangle XDC, \ \triangle ABE \sim \triangle CYE, \quad \text{and moreover} \quad \triangle ABP \sim \triangle CPY.$$

This time, AB is everywhere, so let us write all ratios in terms of AB.

$$\frac{CY}{AB} = \frac{CE}{AE}, \ \frac{CX}{AB} = \frac{CD}{DB} \ \text{ and } \ \frac{XY}{AB} = \frac{CP}{PF}.$$

However, we can use the fact that $CY + CX = XY$. Thus,

$$\frac{CE}{AE} + \frac{CD}{DB} = \frac{CP}{PF}.$$

Solution to Example 4.5.

Note that $\triangle PRQ \sim CBA$. It is enough to find the scale factor between $\triangle PQR$ and $\triangle ABC$ through PQ/CA or PR/CB. Then, we square the length ratio to find the area ratio, as we found in **Example 4.1**.

By Menelaus's, we have

$$\frac{CE}{EA} \cdot \frac{AP}{PD} \cdot \frac{DB}{BC} = \frac{1}{3} \cdot \frac{AP}{PD} \cdot \frac{5}{7} = 1 \implies \frac{AP}{PD} = \frac{21}{5}.$$

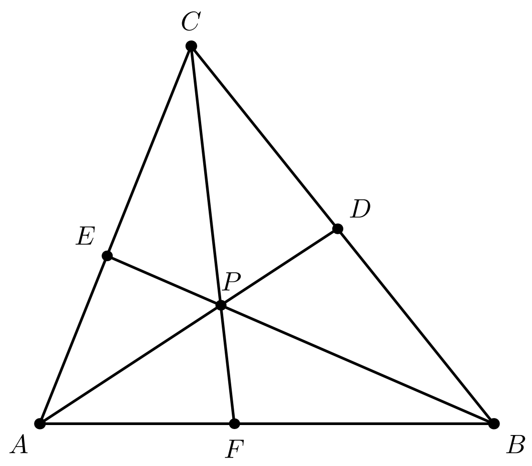
After recognizing $\triangle APR \sim \triangle ADB$, and some algebraic manipulation, we get

$$\frac{AP}{AD} = \frac{PR}{DB} = \frac{21}{26} \implies \frac{PR}{CB} = \frac{PR}{BD} \cdot \frac{BD}{CB} = \frac{21}{26} \cdot \frac{5}{7} = \frac{15}{26}.$$

So, it follows that

$$\frac{[PRQ]}{[CBA]} = \left(\frac{PR}{BC}\right)^2 = \left(\frac{15}{26}\right)^2 = \frac{225}{676}.$$

Solution to Example 4.6.



Let us take $K_a = [BPC], K_b = [CPA], K_c = [APB]$.

Consider the areas of $\triangle AFC$ and $\triangle BFC$. Observe that

$$\frac{[AFC]}{[BFC]} = \frac{AF \cdot h_c}{FB \cdot h_c} = \frac{AF}{FB}.$$

What happened? As both triangles have the same altitude h_c, the altitudes canceled and we are left with the base ratio.

The same can be applied to $\triangle APF$ and $\triangle BFP$. We have

$$\frac{[AFP]}{[BFP]} = \frac{AF}{FB}.$$

A cool trick to know when dealing with ratios is that

$$\frac{a}{b} = \frac{c}{d} = k \implies \frac{a \pm c}{b \pm d} = k.$$

Therefore, we may say that

$$\frac{[AFC] - [AFP]}{[BFC] - [BFP]} = \frac{K_b}{K_a} = \frac{AF}{FB}.$$

We are actually done. Since the above also holds for the other two side ratios. Indeed,

$$\frac{K_c}{K_b} = \frac{BD}{DC} \quad \text{and} \quad \frac{K_a}{K_c} = \frac{CE}{EA}$$

Multiplying the three equations yield

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{K_b}{K_a}\frac{K_c}{K_b}\frac{K_a}{K_c} = 1,$$

which is Ceva's theorem.

This method is powerful, since we related the side ratios with "more general" areas of the triangle. The K's we defined are symmetric, so we only have three such K's, instead of six lengths.

Solution to Example 4.7.

It is likely that we won't be able to solve anything directly. However, maybe we can find some nice relations between the two expressions, by considering K's.

Recall the ratio trick in **Example 4.6**. Notice that

$$\frac{AO}{OA'} = \frac{[COA]}{[COA']} = \frac{[BOA]}{[BOA']} \implies \frac{AO}{OA'} = \frac{[COA] + [BOA]}{[COA'] + [BOA']} = \frac{K_b + K_c}{K_a}$$

It remains to sum the other two up, and

$$\begin{aligned}
\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} &= \frac{K_b + K_c}{K_a} + \frac{K_c + K_a}{K_b} + \frac{K_a + K_b}{K_c} \\
&= \frac{K_b^2(K_a + K_c) + K_c^2(K_a + K_b) + K_a^2(K_b + K_c)}{K_aK_bK_c} \\
&= \frac{K_b^2(K_a + K_c) + K_c^2(K_a + K_b) + K_a^2(K_b + K_c) + 2K_aK_bK_c}{K_aK_bK_c} - 2 \\
&= \frac{(K_a + K_b)(K_b + K_c)(K_c + K_a)}{K_aK_bK_c} - 2 \\
&= \frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} - 2
\end{aligned}$$

where a clever factoring trick was used. Though this isn't as clean as Ceva's theorem, it is still a nice result.

Back to our problem, we simply plug in the two expressions, and our answer then is $92+2=94$.

Solution to Example 4.8. Once again, we want to work with ratios, though nothing is quite given. Here, we must note that O is special because $OA = OB = OC$, by definition of the circumcenter. Therefore, we can throw an R on each term!

$$\begin{aligned}\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF} &= \frac{1}{R}\left(\frac{AO}{AD}+\frac{BO}{BE}+\frac{CO}{CF}\right)\\ &= \frac{1}{R}\left(3-\frac{OD}{AD}-\frac{OE}{BE}-\frac{OF}{CF}\right)\end{aligned}$$

Now what? Look at OD/AD. Suppose that we drop altitudes h_a, h_o from A, O to BC. We have that

$$\frac{OD}{AD}=\frac{h_o}{h_a}=\frac{h_o\cdot BC}{h_a\cdot BC}=\frac{K_a}{[ABC]}$$

This shows us that the ratio of two triangles that share the same base not only has area ratio equal to height ratio, but also a "slanted" height ratio, since we can construct similar right triangles to make things right (kind of like in **Example 4.1**, though here angles are supplementary instead of congruent).

$$\begin{aligned}\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF} &= \frac{1}{R}\left(3-\frac{K_a}{[ABC]}-\frac{K_b}{[ABC]}-\frac{K_c}{[ABC]}\right)\\ &= \frac{1}{R}(3-1)=\frac{2}{R}\end{aligned}$$

So, it remains to find R. We have

$$R=\frac{9\cdot 10\cdot 11}{4[ABC]}=\frac{9\cdot 10\cdot 11}{4\sqrt{15\cdot 4\cdot 5\cdot 6}}=\frac{33}{4\sqrt{2}} \implies \frac{2}{R}=\frac{8\sqrt{2}}{33}.$$

Solution to Example 4.9.

Let us set up $AK=\alpha, BK=\beta$. By the parallel condition, we have $\triangle ABK \sim CDK$, so $KC=\frac{4}{3}\alpha$, $KD=\frac{4}{3}\beta$.

Now, by noticing that $\triangle AKD$ shares heights with $\triangle AKB$ we have

$$\frac{[AKD]}{[AKB]}=\frac{DK\cdot d_{A,BD}}{BK\cdot d_{A,BK}}=\frac{DK}{BK}=\frac{4}{3},$$

where $d_{A,BD}$ - the altitude from A to BD - cancels. Therefore, the area ratio is simply the ratio of the base lengths.

Similarly, we have

$$\frac{[AKD]}{[CKD]}=\frac{AK\cdot d_{D,AC}}{CK\cdot d_{D,AC}}=\frac{AK}{CK}=\frac{3}{4}.$$

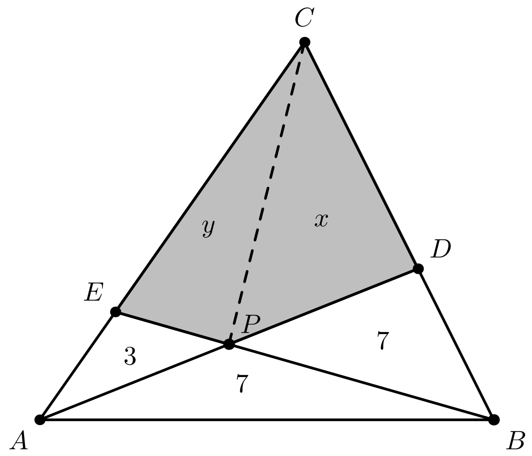
So, $[AKB]=18$, $[CKD]=32$. Finally, $\triangle BKC$ has a same interior angle as $\triangle AKD$, which makes (using the results of **Example 4.1**)

$$\frac{[AKD]}{[BKC]}=\frac{AK\cdot DK}{BK\cdot CK}=\frac{\alpha\cdot\frac{4}{3}\beta}{\frac{4}{3}\alpha\cdot\beta}=1$$

Our final answer is $24 + 24 + 18 + 32 = 98$.

As a corollary, we found that in a trapezoid, $[AKD] = [BKC]$. A slicker way to show this is to realize that $[ACD] = [BCD]$, and then subtract $[KCD]$.

Solution to Example 4.10.



Let us assign letters A, B, C, D, E, P, as indicated. Draw CP, and let $[DCP] = x$, $[ECP] = y$.

Consider AP/PD, which corresponds to two different pairs of area ratios, as we look on both sides of the line.

$$\frac{AP}{PD} = \frac{[ABP]}{[PBD]} = \frac{7}{7} = \frac{[ACP]}{[PCD]} = \frac{3+y}{x}.$$

Likewise for BP/PE, we get

$$\frac{BP}{PE} = \frac{[BAP]}{[PAE]} = \frac{7}{3} = \frac{[BCP]}{[PCE]} = \frac{7+x}{y}.$$

Thus, we have a system of linear equations. This solves for

$$x = \frac{15}{2}, y = \frac{21}{2} \implies [EPDC] = 18.$$

Solution to Example 4.11.

The key to these problems is to split areas up cleverly, and prove that each case is equal. Here, the most logical choice is to construct P on YZ such that $BP \perp YZ$. Now, we try to show that $[ZQPA] = [ABVU]$.

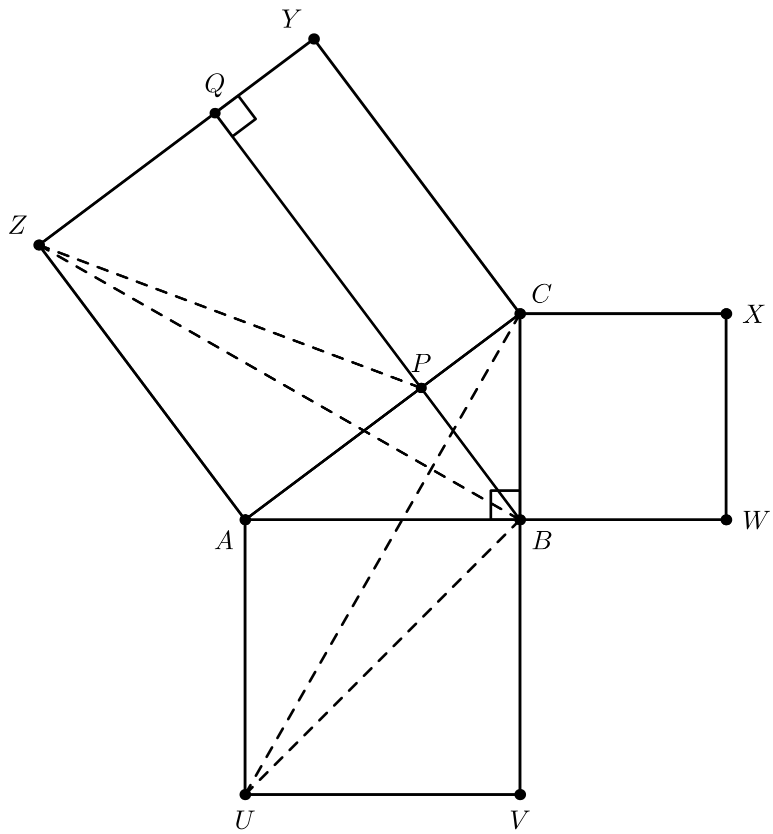
We are familiar with dealing with triangles, so we split the rectangle/square in half. Then, we try to find another triangle of equal area.

$$\frac{[ZQPA]}{2} = [ZPA] = [ZBA]$$

$$\frac{[ABVU]}{2} = [ABU] = [ACU]$$

Now, observe that $AC = AU, AZ = AB$ by construction of squares. Furthermore, $\angle ZAB = 90^\circ + \angle CAB = \angle CAU$. Thus,

$$\triangle ZAB \cong \triangle CAU \text{ and hence } [ZAB] = [CAU].$$

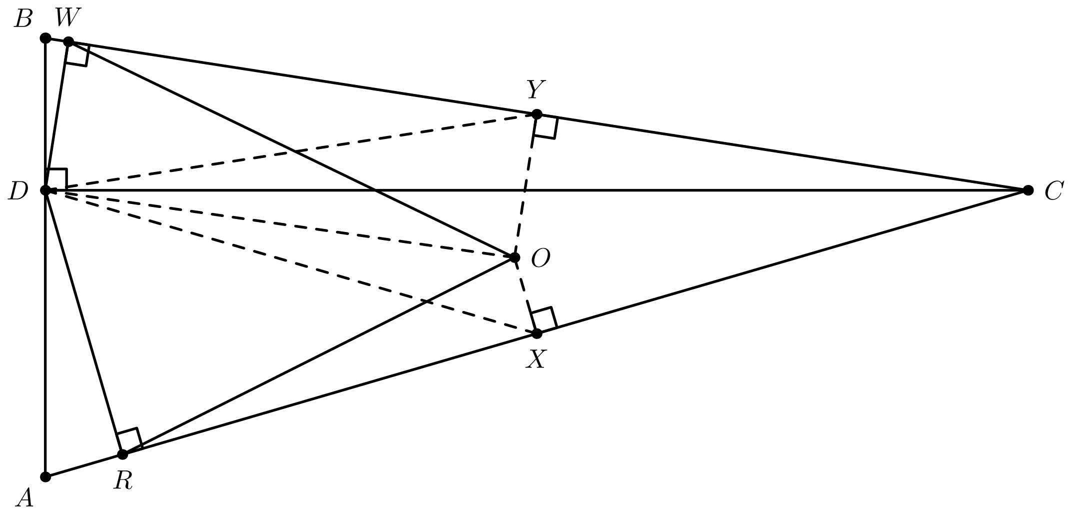


This implies that the area of the rectangle and the square is equal. By repeating this process to the other side of the triangle, we get $[YQPC] = [BCXW]$. So, it follows that

$$[ABVU] + [BCXW] = [ZQPA] + [YQPC] = [ACZY].$$

Solution to Example 4.12.

To use the circumcenter condition, let X, Y be the altitude from O to AC, BC, respectively. We have $OX \parallel DR$, $OY \parallel DW$, so this prompts us to slide some areas.



The natural attempt is to break $[CROW] = [CXOY] + [XOR] + [YOW]$. Note that by our parallel construction,

$$[XOR] = [XOD] \quad \text{and} \quad [YOW] = [YOD].$$

Then, $[CROW] = [CXOY] + [XOD] + [YOD] = [CXDY]$. O conveniently vanishes from our equations!

We are not done yet. Recall that X, Y are midpoints of AC, CB. To use this, we break up the area again with $[CXDY] = [CXD] + [CYD]$.

It is also true that

$$[CXD] = \frac{1}{2}[CAD] \quad \text{and} \quad [CYD] = \frac{1}{2}[CBD],$$

hence $[CXDY]$ has half the of $[ABC]$.

To find $[ABC]$, we can directly use Heron's, and obtain

$$[CROW] = \frac{1}{2}[ABC] = 126.$$

5 Solutions to Section 5 Examples

Solution to Example 5.1.

Despite being not a relatively complicated problem, it is quite easy to make computation errors. So, let's be careful.

We first rewrite the condition in "nicer" angles. We know that $BD \parallel AE \parallel HG$ by symmetry, so

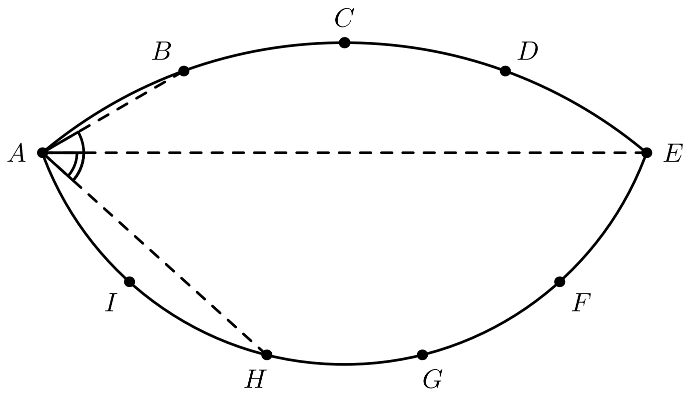
$$\angle ABD = \angle AHG + 12^\circ \iff \angle BAE = \angle HAE - 12^\circ.$$

Now, let us consider the measures of the arcs. Let the arc measures be $\widehat{AB} = x$, $\widehat{AI} = y$. We are also given that all small arcs on the bigger arc have the same measure.

By properties of inscribed angles in circles, we note

$$\angle BAE = \frac{1}{2}\widehat{BE} = \frac{3}{2}x \text{ and } \angle HAE = \frac{1}{2}\widehat{HE} = \frac{3}{2}y.$$

So, it follows that $y = x + 8^\circ$.



Now, we need to use the fact that C is the center of the bottom arc.

$$\angle ACE = \widehat{AE}_{bottom} = 5y$$

Looking back at the upper arc,

$$\angle CAE = \angle AEC = \widehat{AC} = 2x.$$

By using $\angle ACE + 2\angle CAE = 180^\circ$, we arrive at

$$5y + 2 \cdot 2x = 180^\circ \implies x = 20^\circ, y = 28^\circ.$$

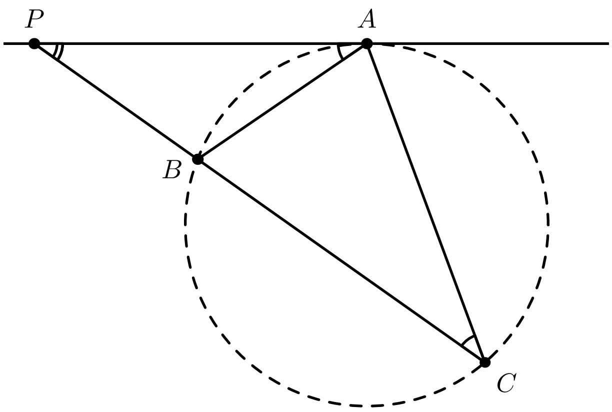
Hence, we can easily compute $\angle BAG$.

$$\angle BAG = \angle BAE + \angle EAG = \frac{3}{2}x + \frac{2}{2}y = 58^\circ.$$

Solution to Example 5.2.

We first need to note an important fact about tangent lines, which we slightly hinted back in **Problem 2.8**.

Suppose $\triangle ABC$ is circumscribed by circle ω. Now, let P be on ray CB so that PA be tangent to ω at A. Recall the property of circles that angles inscribing the same arc have the



same measure. Now, $\widehat{AB}$ is obviously inscribed by $\angle CAB$. However, since PA is tangent, we should also have (intuitively) $\angle PAB$ inscribing $\widehat{AB}$. This implies that $\angle PAB = \angle ACB$.

Another way to think about this is through Power of a Point. Imagine line ℓ through P intersecting ω at A, A'. We have $PB \cdot PC = PA \cdot PA'$. In the limit that ℓ becomes tangent, $A = A'$, and then $PB \cdot PC = PA \cdot PA = PA^2$ should still hold. This rearranges to $PB/PA = PA/PC$, which thereby suggests that $\triangle PBA \sim \triangle PAC$.

Of course, both views really arrive at the same conclusion. For sake of this problem, lengths is what we need.

Returning to the problem, we note that BD is tangent to ω_1, so

$$\triangle DEB = \triangle DBA \text{ and } \frac{EB}{ED} = \frac{BD}{BA}.$$

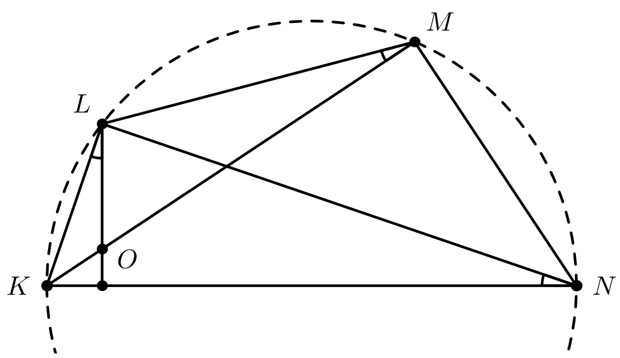
Now, BA is also tangent to ω_2. Taking the power of a point for B with respect to ω_2, we have

$$BA^2 = BD \cdot BC = 3 \cdot 16 = 48 \implies BA = 4\sqrt{3}$$

Therefore, $BD/BA = 4\sqrt{3}/3$.

Solution to Example 5.3.

First, we need to observe that $KLMN$ is cyclic with diameter KN, since $\angle KLN = \angle KMN = 90°$.



The key to this problem is finding the right pair of similar triangles, which isn't too hard if we start labeling congruent angles. We have $\angle LMK = \angle LNK$ by the cyclic condition. Furthermore, $\angle LNK = 90° - \angle LKN = \angle KLO$, which comes from the two right triangles.

So, $\angle LMK = \angle KLO$, which means that by AA,

$$\triangle KLO \sim \triangle KLM.$$

We finish with ratios:

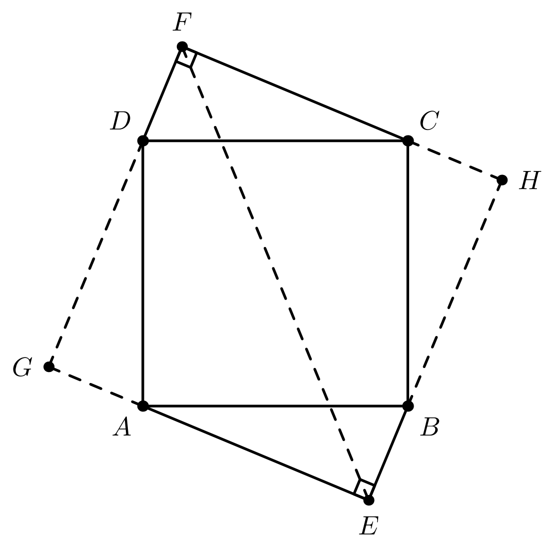
$$\frac{KL}{KM} = \frac{KO}{KL} \implies KM = \frac{28^2}{8} = 98.$$

Our answer is $OM = 90$.

Solution to Example 5.4.

It is not hard to compute this directly, if we drop a bunch of perpendiculars. However, there is a much slicker solution involving symmetry. Construct G, H such that $AG = CH = 5$, $DG = BH = 12$.

Now, the four triangles $\triangle ABE, BCH, CDF, DAG$ are congruent. Moreover, G, A, E and symmetric variations are collinear, since $\angle EAB + \angle BAD + \angle DAG = \angle EAB + 90^\circ + \angle ABE = 180^\circ$.



Therefore, $GEHF$ is a square with side lengths $12 + 5 = 17$. Then, EF, the diagonal, must have length $17\sqrt{2}$.

Solution to Example 5.5.

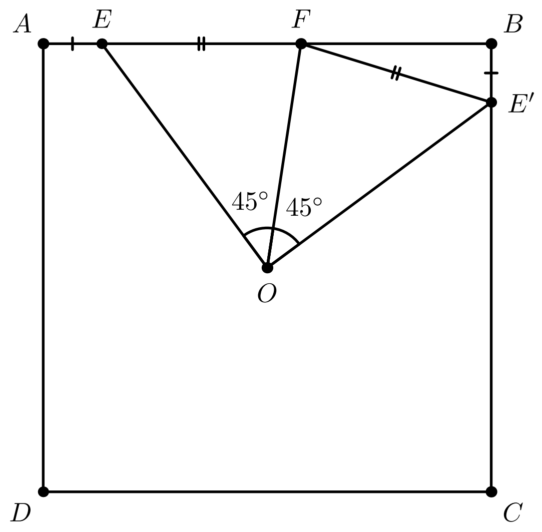
See that there is a 45° angle in the problem. That isn't only used to make our computation easier, but to allow for symmetry.

Rotate E 90° about O to E' onto BC. By symmetry of the square, $AE = BE'$. Also, $\triangle OFE \cong OFE'$, since $\angle FOE = \angle FOE' = 45^\circ$, and $OE = OE'$.

Now, the congruence condition gives us $FE' = FE = 400$, which is enough to finish. Let $BF = x$, then $BE' = AE = 500 - x$. We have

$$FB^2 + BE'^2 = FE'^2 \text{ or } x^2 + (500 - x)^2 = 400^2$$

Solving gives $BF = x = 250 + 50\sqrt{7}$, as we take the larger solution.

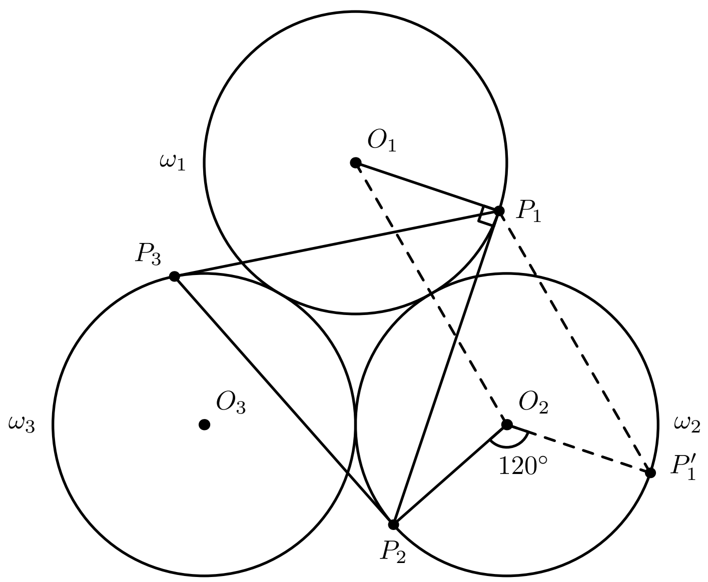


Solution to Example 5.6.

It is quite unclear how to approach the problem. All points P are defined "relative" to another, so we don't have anywhere to start. Fortunately, we take make use of the symmetry of the problem, and demand nice things to happen.

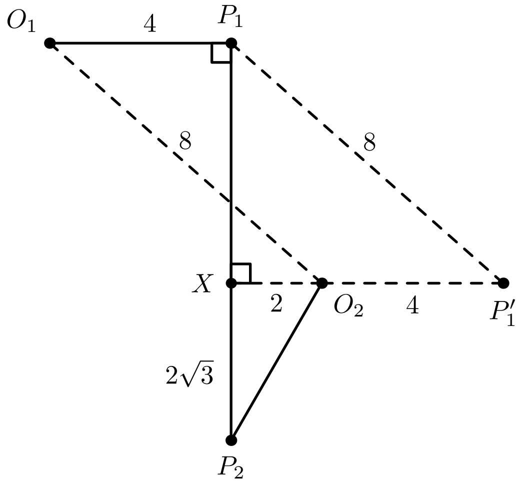
Suppose let us translate P_1 onto ω_2, keeping the same "angle" with respect to the center of the circles. Of course, $O_1O_2P_1'P_1$ is a parallelogram, by construction. However, we can use the symmetry of the problem to demand that

$$\angle P_2O_2P_1' = 120^\circ.$$



It turns out that that is all we need! We know that $P_1P_2 \perp P_1P_2$, $O_1P_1 \parallel O_2P_1'$, and $\angle P_1P_2O_2 = 30^\circ$. It makes sense to extract this part of the figure out, where we can more clearly visualize.

We also drop the perpendicular from O_2 to P_1P_2, with foot at X. Now, $\triangle XP_2O_2$ is a 30-60-90 triangle, and $XP_2 = 2\sqrt{3}$, $XO_2 = 2$. We also had $O_1O_2 = P_1P_1' = 8$ and $O_1P_1 = O_2P_1' = 4$.



To find P_1P_2, we need P_1X, and with the Pythagorean Theorem

$$P_1X = \sqrt{P_1P_1'^2 - XP_1'^2} = \sqrt{8^2 - 6^2} = 2\sqrt{7}$$

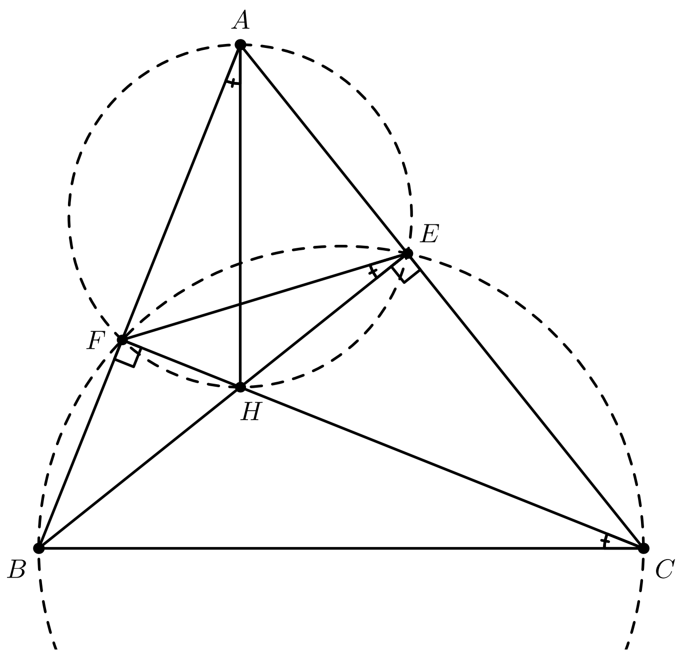
Finally, $P_1P_2 = 2\sqrt{7} + 2\sqrt{3}$, and the area of equilateral $\triangle P_1P_2P_3$ is

$$\frac{\sqrt{3}}{4}\left(2\sqrt{7} + 2\sqrt{3}\right)^2 = \sqrt{300} + \sqrt{252}$$

Solution to Example 5.7.

Let BE, CF intersect at H. It suffices to show that $\angle BAH = 90° - \angle B$, so H, D will be collinear. The two altitudes gives us two nice cyclic quadrilaterals.

$\angle BFC = \angle BEC = 90°$ implies $BFEC$ is cyclic. Also, $\angle AFH = \angle AEH$ implies $AFHE$ is cyclic.



The rest is straightforward, as we just switch to the other angle in the cyclic quadrilateral inscribing the same arc (there is only one). Letting $BFEC = \omega_1$, $AFHE = \omega_2$:

$$\angle FAH \overset{\omega_2}{=} \angle FEH \overset{\omega_1}{=} FCB$$

However, $\angle FCB = 90 - \angle B$, and then so is $\angle FCB$. Therefore, H must lie on the altitude of A, as desired.

As an important corollary, we can also note that $\triangle AEF \sim \triangle ABC$, since $\angle ABC = 180° - \angle FEC = \angle AEF$.

Solution to Example 5.8.

First, we have a 13-14-15 triangle. It is well-known that the altitude to the 14 side is 12. Indeed, we can see through Heron's that

$$A = \sqrt{21 \cdot 6 \cdot 7 \cdot 8} = 84 = \frac{14 \cdot 12}{2}.$$

Furthermore, note that

$$\sqrt{13^2 - 12^2} = 5 \text{ and } \sqrt{15^2 - 12^2} = 9$$

(Indeed, the 13-14-15 triangle can be decomposed into a 5-12-13 triangle and a 9-12-15 triangle by the altitude)

Looking at $\triangle ADC$, it is clear that $\angle ADE = \angle C$, and $\triangle ADE \sim \triangle ACD$. Also, DE is the altitude, which we can easily compute as $DE = 9 \cdot 12/15$.

Since $\angle ADB = \angle AFB = 90°$, we immediately note that $AFDB$ is cyclic. Remember that $\angle ADF = \angle C$, so by congruent angles in a cyclic quadrilaterals we have

$$\angle ABF = \angle ADF = \angle C.$$

Now, since $\angle ABF = \angle ACD$ and $\angle AFB = \angle ADC$, we have

$$\triangle ABF \sim \triangle ACD.$$

Now, we are able to compute both AF and AE with ratios, and hence EF by Pythagorean Theorem.

$$AF = \frac{AD}{AC} \cdot AB \text{ and } AE = \frac{AD^2}{AC}.$$

$$EF = \sqrt{AF^2 - AE^2} = \frac{AD}{AC} \cdot \sqrt{AB^2 - AD^2} = \frac{12}{15} \cdot 5.$$

Our answer is finally

$$DF = DE - EF = 9 \cdot \frac{12}{15} - 5 \cdot \frac{12}{15} = \frac{16}{5}.$$

Solution to Example 5.9.

There are two cases of this problem, for whether P lies on arc $\widehat{AB}$ or $\widehat{BC}$. Let us suppose that P is on $\widehat{AB}$, though in the other configuration the derivation is similar.

Q is floating outside of ABC somewhere in the open. Q is defined in terms of BP and DF, so we obviously need to use some angle features of these two lines.

Thinking of angles, really the only thing we can hope for is that somewhere the two lines create congruent angles, so we may form a cyclic quadrilateral. Ideally, the cyclic quadrilateral should contain A, so we actually end up with AQ.

It isn't too hard to guess that $AFPQ$ is cyclic. Indeed, we will show that

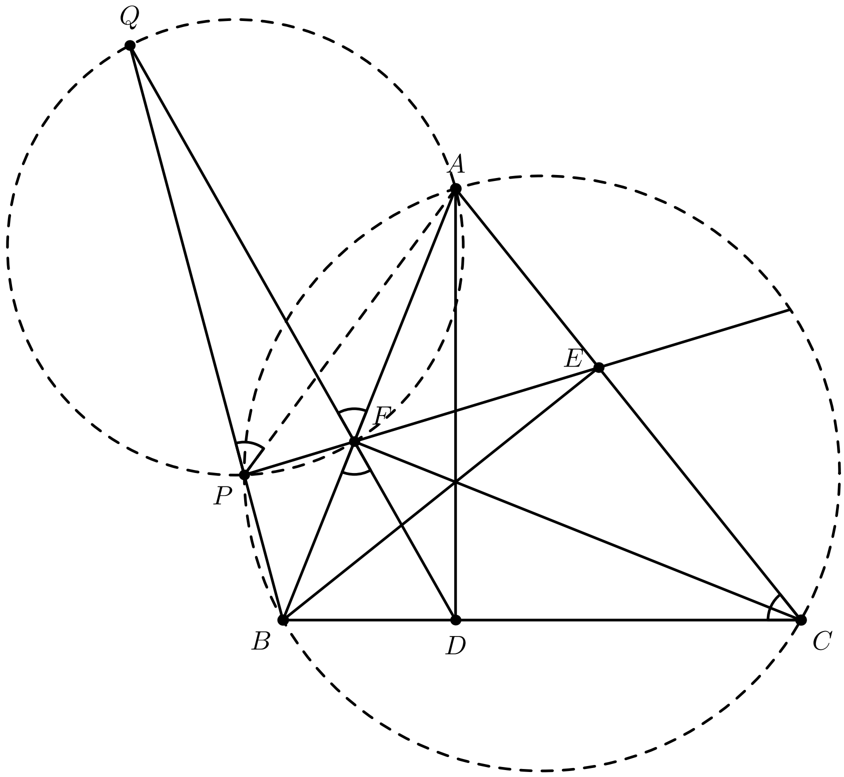
$$\angle AFQ = \angle C = \angle APQ.$$

The fact that opposite angles in a cyclic quadrilateral are supplementary gives

$$\angle C \overset{\odot ABC}{=} 180^\circ - \angle APB = \angle APQ.$$

Here, we used that Q lies on BP.

Now, remember that $\triangle BDF \sim \triangle BAC$ (from **Example 5.7**), we get $\angle C = \angle BFD = \angle AFQ$, where the later equality follows from Q being on DF.



Nice, so $AFPQ$ is cyclic. Now, we want $AQ = AP$, or $\angle QAP = \angle PAQ = \angle C$. We use the cyclic condition to transform

$$\angle QAP \overset{AFPQ}{=} 180^\circ - \angle AFP.$$

It remains to show $AFP = 180^\circ - \angle C$. But this is true because $\angle AFE = \angle C$, as $\triangle AFE \sim ACB$. Here, we used the similarity again, but with triangles at vertex A.

So, we are done. The other configuration (where P is on the other side of EF) is similar.

Solution to Example 5.10.

Since this is literally a synthetic geometry handout, our first instincts (hopefully) are to look for synthetic observations.

By the reflection condition, $\angle ABD = \angle ABC$. However, note that $\angle ABC = \angle ACB$. As $\angle ACB$ inscribes arc $\widehat{AB}$, so must $\angle ABD$.

But then this must mean that DB is tangent to $\odot ABC$ (from **Example 5.2**)!

The rest is algebra and some Pythagorean Theorem. We use the circumcenter, and note that $\angle OBD = 90^\circ$. So,

$$OC = OB = \sqrt{OD^2 - OB^2} = \sqrt{2}.$$

The distance from O to BC is

$$\sqrt{OB^2 - \left(\frac{BC}{2}\right)^2} = \frac{\sqrt{2}}{2} \implies h_a = 1 + \frac{\sqrt{2}}{2}.$$

Finally, the area is

$$\frac{1}{2} \cdot h_a \cdot BC = \frac{1}{2} \cdot \left(1 + \frac{\sqrt{2}}{2}\right) \cdot \sqrt{2} = \frac{\sqrt{2}+1}{2}.$$

By the way, we did assume that $\triangle ABC$ is acute. If not, we would get a slightly different answer. (So yes, this problem does have configuration issues)

Solution to Example 5.11.

First, we show that the reflection of H over BC, H', is on $\odot ABC$.

Our strategy is instead to first let AD intersect $\odot ABC$ at H', and argue $HD = H'D$, since we like angles more.

Note that since H' lies on $\odot ABC$ (by definition), we get

$$\angle DCH = \angle 90^\circ - \angle B = \angle BAD \overset{\odot ABC}{=} \angle DCH'$$

where the last equality uses congruent angles in the cyclic quadrilateral.

Similarly, we get $\angle DBH = \angle DBH'$. Thus, $\triangle HBC \sim \triangle BH'C$ by SAS, and it follows that $HD = HD'$.

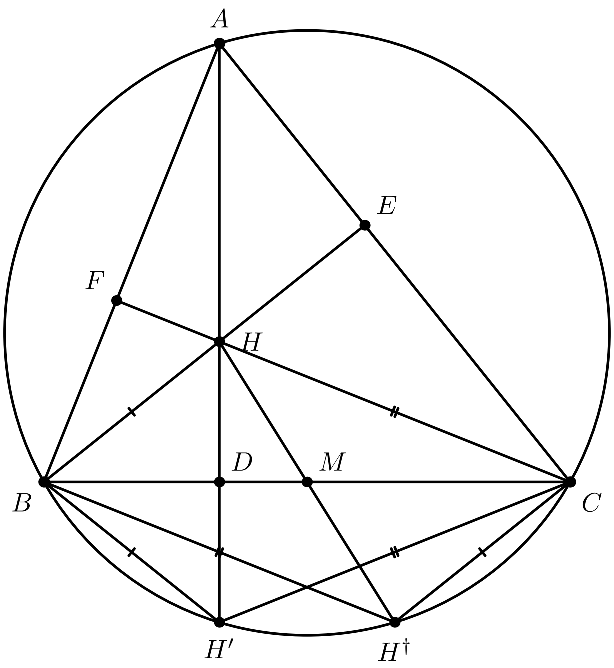
Now, to show that H reflected over M, $H^\dagger$ lies on $\odot ABC$, the same approach doesn't work. The line HM is not nice, as far as we can tell, since M embeds a length condition instead of an angle condition.

We use a different strategy. Remember when segments are bisected, we can use a parallelogram to "switch" the congruence condition? Here, we aren't really switching, but we may demand that $BHCH^\dagger$ is a parallelogram because both segments are by construction bisecting each other.

This gives us $\triangle BHC = CH^\dagger B$. Now, doesn't that look familiar? This is the same as H', except reflected!

To finish, we may say that $BHH^\dagger C$ is an isosceles trapezoid, so $BHH^\dagger C$ is cyclic. Hence, $H^\dagger$ lies on $\odot ABC$.

As an extra note, it is interesting that H', $H^\dagger$ are on opposite sides of $\widehat{BC}$, hence on opposite sides of the angle bisector of $\angle BAC$. Interestingly, the orthocenter H and the circumcenter O are also opposite of each other with respect to angle bisectors. (Can you show $\angle BAH = \angle CAO$?) As a consequence, O lies on $AH^\dagger$.

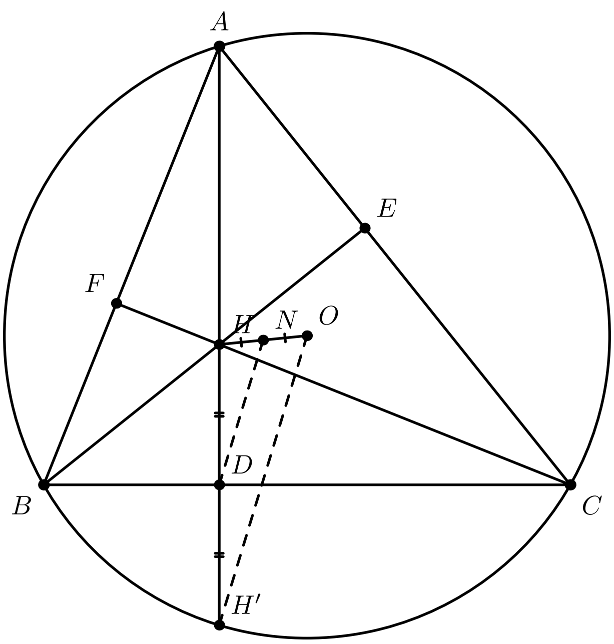


Solution to Example 5.12.

First, we wonder, what is nice about these points? Remembering the previous example, these points lie "halfway" between the orthocenter H and the circumcircle. This must lead us somewhere.

Indeed, we want to create more bisected segments. Why not try the midpoint of H and O? Let this point be N. Suppose that some point X has the property that H reflected over X to H' lies on $\odot ABC$. (Indeed, all 9 points we have do satisfy this property). Then,

$$HX = \frac{1}{2}HH' \text{ and } HN = \frac{1}{2}HO.$$



Now, this is enough to imply $\triangle HNX \sim HOH'$. We consider the third pair of segments, and

$$NX = \frac{1}{2}OH' = \frac{1}{2}R.$$

This is really nice! H is removed from the equation, and OH' is merged into R. Now, NX is constant, so all points X lie on a circle centered at N with radius $\frac{R}{2}$.

Indeed, this is the 9-point circle. The center N_9 of the 9-point circle is the midpoint between the orthocenter and the circumcenter, and the radius of the 9-point circle is half that of the circumcircle.

Solution to Example 5.13.

First, We need to obtain more information of I in hopes of getting something useful. Draw AI, and AI must be the angle bisector of $\angle BAC$.

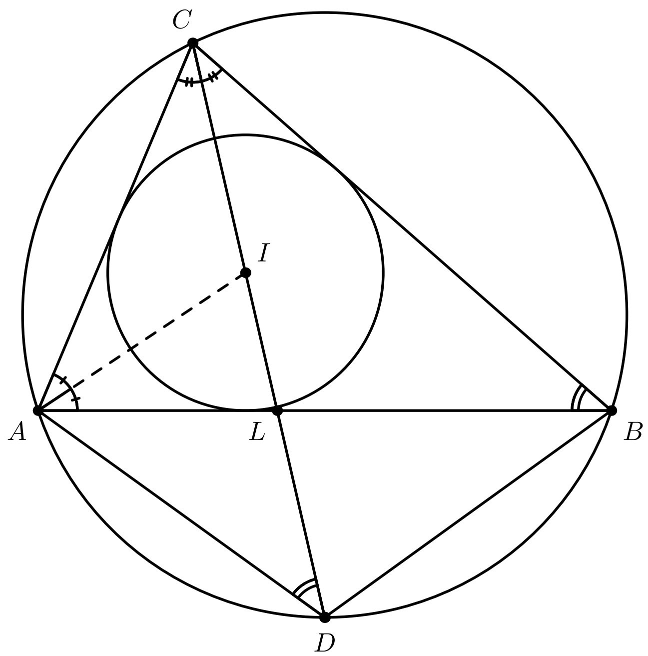
$$\angle CIA = 180^\circ - \angle ICA - \angle IAC = 180^\circ - \frac{1}{2}\angle C - \frac{1}{2}\angle A = 90 + \frac{1}{2}\angle B$$

Then, we also have

$$\angle AID = 180^\circ - \angle CIA = 90 - \frac{1}{2}\angle B$$

How interesting. Remember way back in **Example 1.2** on what we gave for angles in an isosceles triangle. This angle would be the base of an isosceles triangle if we have an $\angle B$ somewhere.

But wait. To use the fact that D is on $\odot ABC$, we have $\angle CDA = \angle CBA = \angle B$. So, $\triangle ADI$ must be isosceles!



Now, $DI = DA$, but how can we use this? Remember that D is the midpoint of $\widehat{AB}$, so also $DB = DI$!

We finish with similar triangles and the angle bisector theorem. Note that

$$\frac{BD}{BL} = \frac{AC}{AL} = \frac{CI}{IL}.$$

Thus, $IC = 10/3$, and we are done.

Solution to Example 5.14.

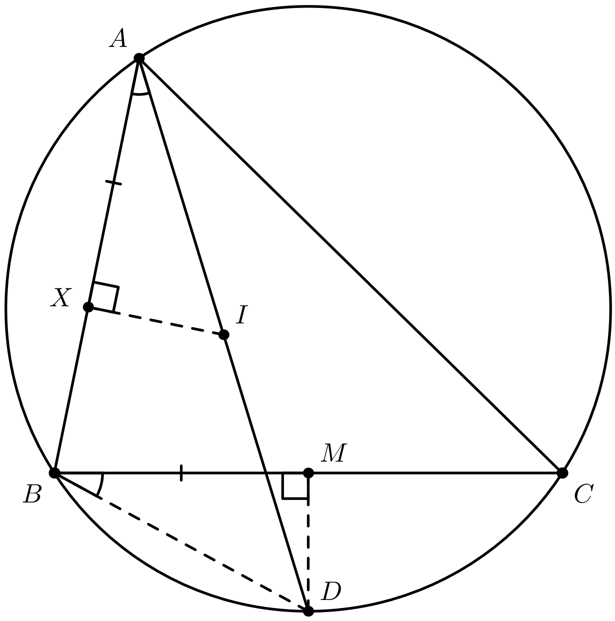
The first question is of course how to use the condition $2BC = AB + AC$. Remember **Problem 1.7** or **Example 3.13**? We showed that the length of tangents from A to the incircle is $s - a$. In our case, $s - a = a/2$, which is certainly something to work off of.

Let X be the point of contact between the incircle and AB. Also, let M be the midpoint of BC. We have $AX = BM = a/2$.

Let us also have AI intersect $\odot ABC$ at D. Now, instead to show $AI = ID$, we can instead try $AI = BD$, using the result from **Example 5.13**.

Seeing these congruences, can't we find a pair of congruent triangles? Indeed, note that $\angle NBD = \angle XAI = \angle A/2$. Furthermore, $\angle AXI = \angle DMB = 90°$. So,

$$\triangle AXI \cong DMB \text{ by ASA!}$$



Then, $AI = BD$, as desired.

Solution to Example 5.15.

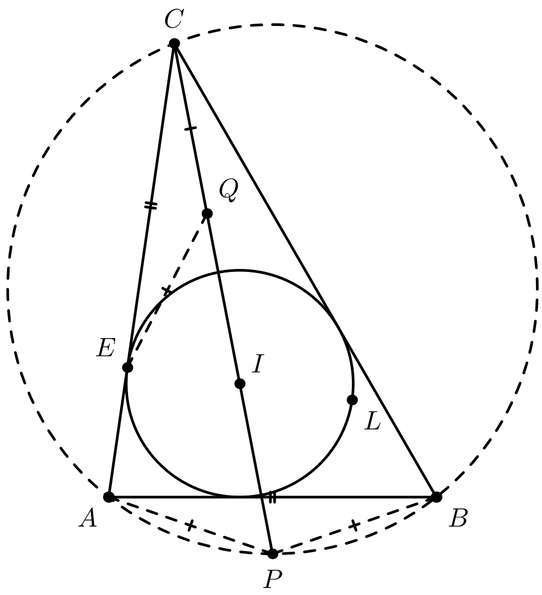
Of course, we have another similar problem, but the length condition slightly changed. This time, note that $s - c = c$.

We start with the incircle, but we aren't quite sure how the reflected points play in. We could, however, define P again as the intersection between CI and $\odot ABC$. We know that A, B, I lies on a circle centered at P. Maybe the same is true for K, L?

Backwards thinking gets us this far. Now, why don't we try to force the congruence again?

Since $EC = AB$ and $\angle ECI = \angle ABD$, we can find somewhere on CI, call it Q, where $\triangle CQE = BDA$.

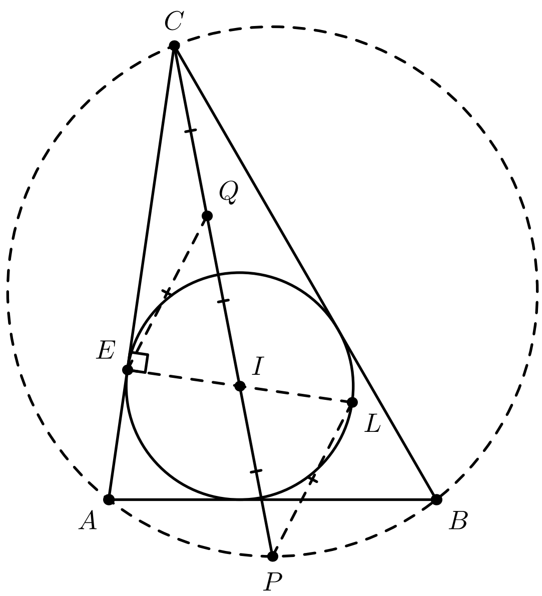
However, $\triangle CEI$ is right, which means that if $QC = QE$, then also $PC = QE = QI$!



That is it. We get $QI = QC = BD = PI$.

So, we have a nice symmetry. L is E reflected over I, while Q is P reflected over I. I bisects both QP and EL, so $PLQK$ is a parallelogram.

It follows that $PL = QE = PI$. The same argument can be made for K, and thus $PA = PK = PI = PL = PB$, and we are done.



Solution to Example 5.16.

We don't see how $R-2r$ can be anything meaningful, so let us instead rearrange the equation into

$$R^2 - OI^2 = 2Rr.$$

That is much better. Remember Power of a Point and **Example 2.7**? We have shown that

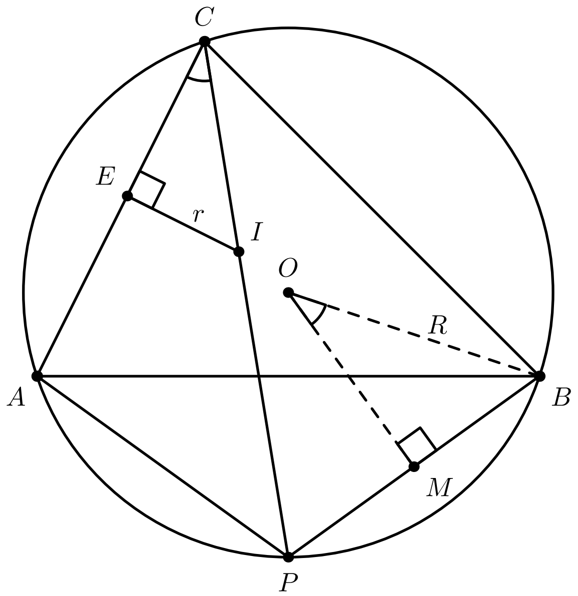
$R^2 - OX^2$ is the power of X with respect to the circle, and can be seen by considering the diameter.

Now, using power of a point, we can transfer the problem statement into any other line through I. Of course, we choose CI, the angle bisector of $\angle ACB$.

Let CI intersect $\odot ABC$ at P. We want to show that

$$CI \cdot IP = 2Rr.$$

This calls for similarity. In fact, we can again use $\triangle CIE$ for CI/IE or CI/r, where E is the point of contact between the incircle and AC.



Now, we need another right triangle with an angle of $\angle C/2$ for similarity. Our constructions in previous problems don't work, since it doesn't contain R.

However, there is a much more obvious choice, and that simply is $\triangle AOP$, or half of $\triangle AOP$. Here, we use $IP = BP$.

Let M be the midpoint of BP. We have $OM \perp BP$ and $\angle BOM = \angle C/2$.

It follows that $\triangle OBM \sim \triangle CIE$ by AA. So,

$$\frac{OB}{BP/2} = \frac{CI}{IE} \implies CI \cdot CP = 2rR.$$

Rearranging everything back, we obtain Euler's formula:

$$IO = R(R - 2r).$$