Ideas and Insight in Synthetic Geometry: Solution to Problems

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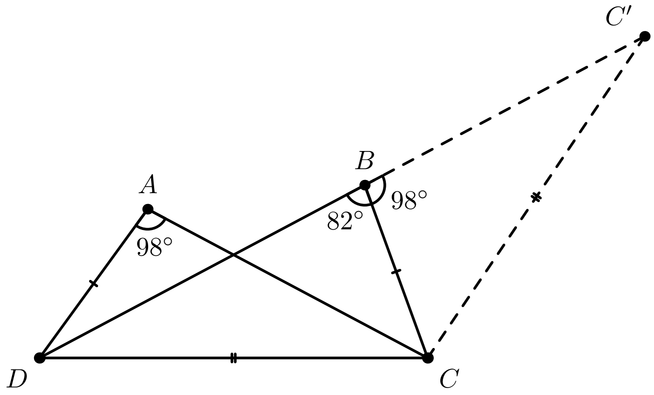
1 Solutions to Section 1 Problems

Solution to Problem 1.1.

Let us try to use the condition of $AD = BC$ in the most obvious way - that they can overlap with each other!

Move $\triangle ADC$ to $\triangle A'D'C'$ where $A'D'$ overlaps with BC. (Alternatively, you may say that you constructed a congruent triangle instead of move, but I think move here better illustrates the motivation)

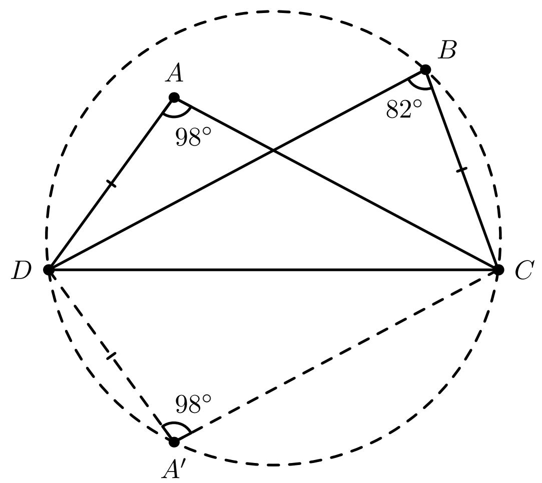


Now, since $\angle DBC$ and $\angle ADC = \angle A'D'C'$ are supplementary, DBC' is actually a straight line!

Now, recall that $D'C'$, or CC' is congruent to DC, since we moved the triangle over. Then, the larger triangle $\triangle DCC'$ is isosceles. This tells us that $\angle BDC = \angle BC'C = \angle ACD$.

Finally, our answer is

$$\angle ACD = \angle BDC = 180^\circ - \angle DBC - \angle DCB = 180^\circ - 82^\circ - 70^\circ = 28^\circ.$$

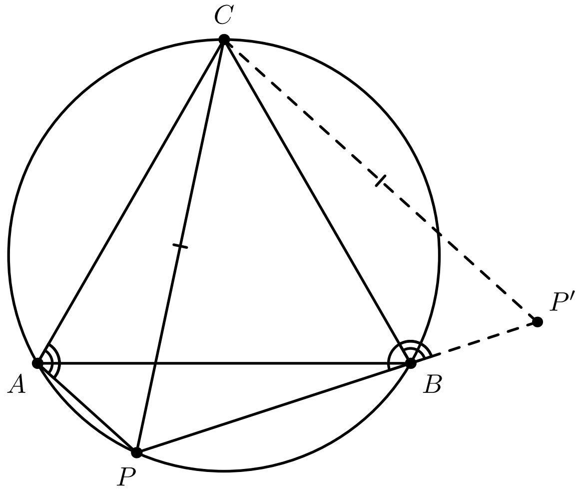


Alternatively, let us try to use the fact that $\angle DAC$, $\angle DBC$ are supplementary to create cyclic quadrilaterals.

Reflect A over CD to A'. Then, $BCA'D$ is cyclic. We may then use $DA' = BC$ to say that the inscribed angles are congruent, that is, $\angle DCA' = \angle CDB = 28^\circ$.

Finally, one more way to think about this problem is that $\triangle CDA$ and CDB are two triangles that are "angle-side-side congruent", or not quite, but they will be guaranteed to have a pair of supplementary (or congruent) angles.

Solution to Problem 1.2.



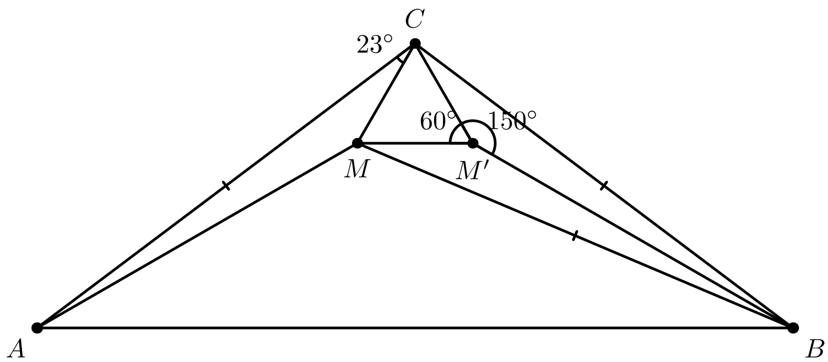
Once again, we are given congruent segments, namely CA and CB. Indeed, let us rotate $\triangle CAP$ by $60°$ about C so that CA' overlaps with CB.

Since $\angle CAP$ and $\angle CBP$ are supplementary, and $\angle CBP' = \angle CAP$, we have that $\angle PBP' = 180°$, so P, B, P' are collinear.

But then, remember that we sent CP to CP', and $\angle PCP' = 60°$, so $\triangle PCP'$ is equilateral! So, we get $CP = CP' = PP' = PB + BP'$, which then gives $CP = AP + BP$.

To recap, there were two important nice events that enabled the construction - that $\angle ACB = 60°$, and that P lied on the circumcircle of $\triangle ABC$, which made P, B, P' collinear.

Solution to Problem 1.3.



First, we label in $\angle ACM = 23°$, $\angle MAC = 7°$. Also, since $ACB = 106°$, $\angle MCB = 83°$. It appears that we can make an equilateral triangle.

Indeed, reflect M' over the line of symmetry of the isosceles triangle. Then, $MCM' = 106° - 2 \cdot 23° = 60°$.

That was nice, but not enough. $7°$ must also play into the diagram nicely. There are quite a few ways to approach the next step, but if you have labeled in $\angle AMC = BM'C = 150°$, then you might have noticed something.

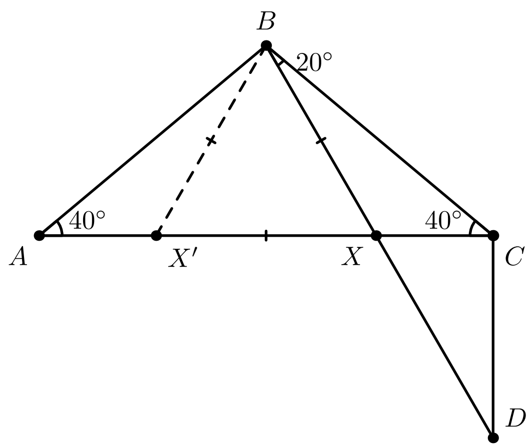
Recall that $\angle MM'C = 60°$, so $\angle MBM' = 360° - 60° - 150° = 150°$. Great, we have $\angle MM'B = CM'B$.

Finally, we make the critical construction that $\triangle M'MB \sim \triangle CMB$ by SAS, so $BM = BC$ and $\angle BMM' = 23°$. It follows that $\angle CMB = \angle BCM = 83°$.

Solution to Problem 1.4.

This nice little problem has two ways to approach. We first present the more straightforward, but maybe just a bit less elegant solution.

Take X to be the intersection of AC and BD. Observe that $\angle ABD - \angle CBD = 60°$, so let us draw X' on AC by reflecting X over the line of symmetry of $\triangle ABC$.



Then, $\angle ABX' = \angle CBX = 20°$, so $\angle X'BX = 60°$. Furthermore, since $BX = BX'$, we have that $\triangle XBX'$ is equilateral.

To use the congruent segment condition, subtract $BX = XX'$ from $AC = BD$ to obtain

$$BD - BX = AC - XX' \implies XD = AX' + XC \implies XD = 2XC$$

However, remember that $\angle DXC = \angle X'XB = 60°$, by our equilateral triangle condition. Thus, $\triangle DXC$ must be a 30-60-90 triangle! It follows that $\angle XCD = 90°$, and $\angle BCD = 40° + 90° = 130°$.

The second approach is to use the congruent segment condition to overlap. Conveniently, we can set up another equilateral triangle.

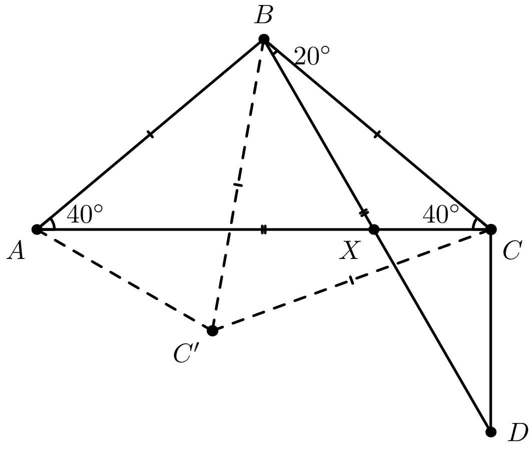
Construct $\triangle B'C'D'$ such that $\triangle B'C'D' \cong \triangle BCD$ and $B'D'$ overlaps with CA. Here,

$$\angle AC'C = \angle DBC = 20° \implies \angle BCC' = 20° + 40° = 60°.$$

So, $\triangle BCC'$ is equilateral, and $BC' = BC = AB$. Therefore, $\triangle ABC'$ is also isosceles!

Then, we find that

$$\angle ABC' = 100° - 60° = 40° \implies \angle BC'A = \frac{180° - 40°}{2} = 70°$$

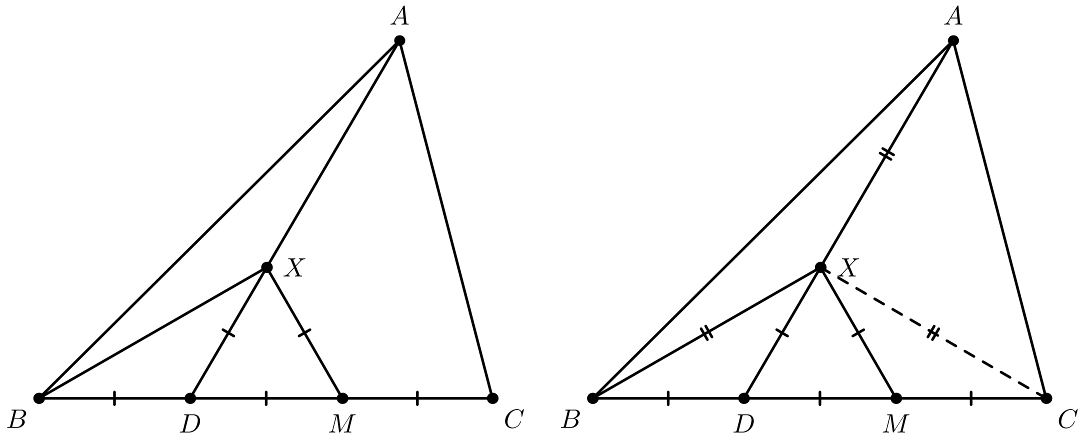


Finally,

$$\angle BCD = \angle CC'A = \angle CC'B + \angle BC'A = 60^\circ + 70^\circ = 130^\circ$$

which agrees with our first method.

Solution to Problem 1.5.



While I feel the above diagram is already a really beautiful proof without words, it still helps to get some idea of the motivation.

There isn't one, so to speak. Our strategy is just guess stuff and hope things like this problem works out nicely.

To start, given a 60° angle, our intuition is immediately to create equilateral triangles. To do this, we consider breaking DC up into DM, MC so that from $DC = 2 \cdot BD$ follows $BD = DM$. Then, we draw equilateral $\triangle DXM$ with X on AD, and $BD = DX$.

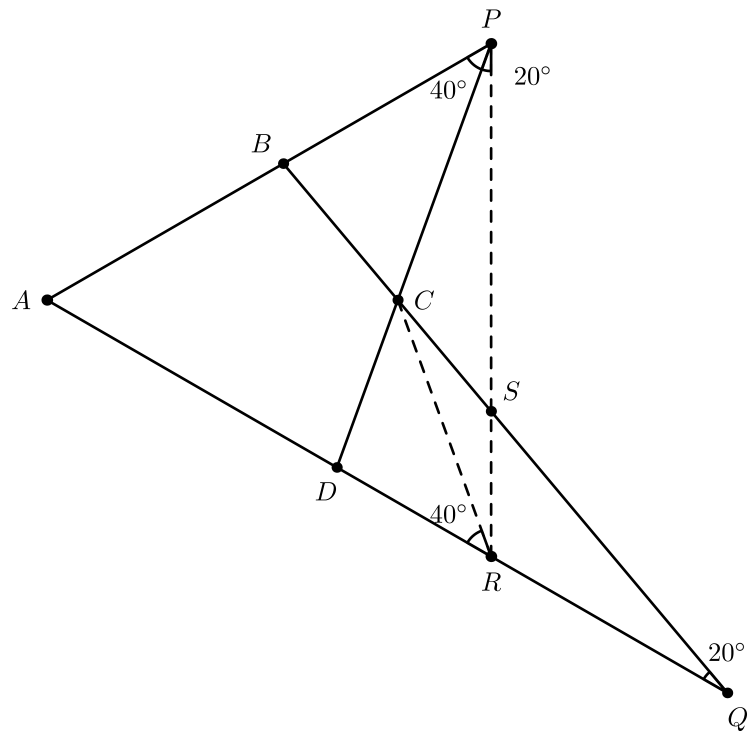
Now, the real coincidence is that

$$\angle BDX = 120^\circ \implies \angle XBD = 30^\circ \implies \angle ABX = 45^\circ - 30^\circ = 15^\circ.$$

But we also have that $BAX = 15^\circ$, so $XB = XA$!

The rest is filling in more tiles of isosceles triangles. $XC = XB$ by symmetry, so $XA = XC$. Then, we calculate some angles to find that $\angle ACB = 75^\circ$.

Solution to Problem 1.6.



We start with $\angle BCD = 180° - \angle BAD = 120°$, and $\angle BDC = \angle DBC = 30°$. So,

$$\angle APD = 180° - \angle PBC - \angle PCB = \angle ABC + \angle BCD - 180° = 70° + 30° + 120° - 180° = 40°.$$

Similarly, we have $\angle AQB = 20°$.

As usual, our instinct is to draw an equilateral triangle. Since $PAQ = 60°$, let us take R on AQ such that $\angle APR = \angle ARP = 60°$.

We start by noting that C lies on the line of symmetry of isosceles $\triangle RAP$ (of course, equilateral implies isosceles). This is because $\angle CAB = \angle CAD = 30°$.

Therefore, by symmetry, we get $\angle DRC = BPC = 40°$. The critical observation is that

$$\angle CDR = 180° - 80° = 100°.$$

So, $\triangle CDR$ must be isosceles!

The rest falls in place. As $\angle DPR = \angle DQC = 20°$, $DR = DC$, and of course $\angle PDR = \angle QDC$, we have that $\triangle DPR \cong \triangle DQC$.

Finally, $\triangle PDQ$ is isosceles, so $\angle DPQ = \angle DQP = 40°$, and our answer is $\angle APQ - \angle AQP = 80° - 40° = 40°$.

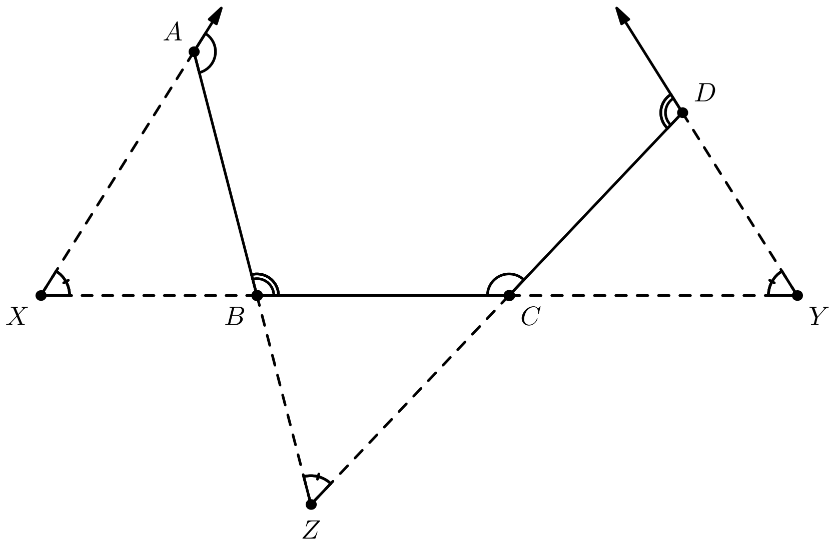
Solution to Problem 1.7.

We have a bunch of congruent segments and angles. How should we make congruent triangles? There is but one obvious approach.

Extend EA to meet BC at X. Similarly, extend ED to meet BC at Y. Finally, let AB and CD meet at Z.

Now, our congruent angles actually bend into congruent triangles. We have, by ASA,

$$\triangle AXB \cong \triangle CZB \cong \triangle CYD.$$

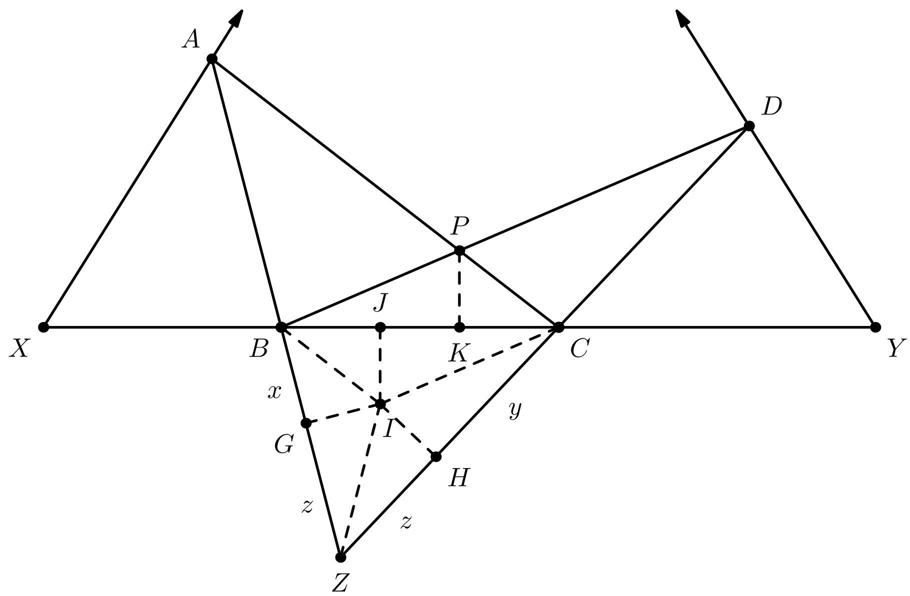


Now, $\angle AXB = \angle CYD$, so $\triangle XEY$ is actually isosceles! This encourages us to simplify the condition: the perpendicular from E to BC is actually the perpendicular bisector of XY. We want to show that the intersection of AC and BD lies on this perpendicular bisector.

Let $\angle BCZ = \alpha, \angle CBZ = \beta$, we can compute, via isosceles triangles $\triangle ABC$ and $\triangle BCD$, that

$$\angle CBD = \frac{180 - \angle BCD}{2} = \frac{\alpha}{2} \quad \text{and} \quad \angle BCA = \frac{180 - \angle CBA}{2} = \frac{\beta}{2}$$

Now, this reminds us of angle bisectors. Why don't we wrap everything inside $\triangle BZC$? If P is the intersection of AC and BD, and I is the incenter of BZC, then $BICP$ is a parallelogram.



Dropping perpendiculars from the incenter is easy. Recall that in **Example 1.1**, we found three pairs of congruent segments. Here, they are in the diagram.

Take $BJ = BG = x$, $CJ = CH = y$, and $ZG = ZH = z$. By $\triangle BPC \cong \triangle CIB$, we have $BK = y$ and $CK = x$.

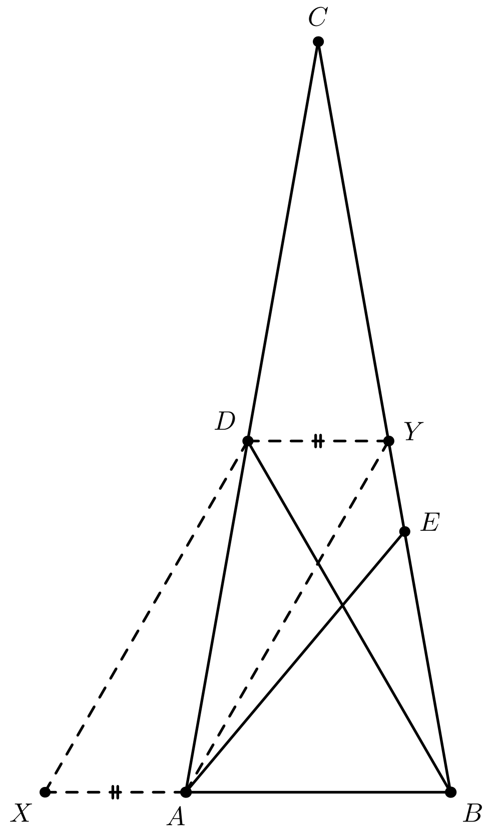
To show that P lies on the perpendicular bisector of XY, it suffices to show that

$$XK = XB + BK = YC + CK = KY$$

However, we know that $XB = BZ = x + z$ and $YC = CZ = y + z$, so the equation is true! We end up with $XK = KY = x + y + z$.

Therefore, we have successfully shown that the three given lines are concurrent.

Solution to Problem 1.8.



This is for sure the toughest problem for chapter 1, but luckily there are also many ways to solve this problem.

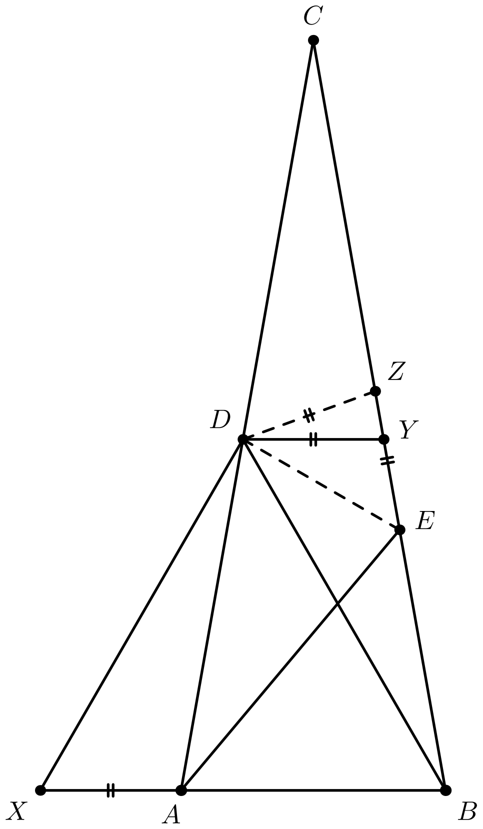
To start, we first observe that $\triangle ABE$, $\triangle BDC$ are isosceles. We don't have any immediate information about anything related to line DE, so lets just try some stuff, and hope we'll get somewhere.

We stick to our ideas, and apply ideas directly. There is a 60° standing out, so let us draw X on AB such that $\triangle XDB$ is equilateral. Also, let us try reflecting D over the line of symmetry of$\triangle ACB$ to get Y.

Now, it certainly appears that $DY = XA$. Indeed, we can prove that $XAYD$ is a parallelogram. By construction, DY is parallel to AB. Furthermore, $\angle YAB = \angle DBA$ by symmetry, and $\angle DBA = \angle BXD \implies XD \parallel AY$.

That is about as far as we can go with these two constructions. However, we still haven't used that $\triangle XDB$ is equilateral (we only needed it being isosceles so far), neither have we used the two isosceles triangles noted in the beginning.

There is now one magical point floating around the diagram that can use all these conditions. We draw Z on BC where $BZ = BD$!



By our construction, observe that

$$EZ = BZ - BE = BD - AB = AX.$$

where we used $\triangle ABE$ and XBD being isosceles.

However, also note that $BD = DC \implies BZ = BD = DC = CY$. Furthermore, since $\angle DCY = \angle DBZ = 20°$, we obtain

$$\triangle DBZ \cong \triangle DCY \implies DZ = DY.$$

Indeed, it happens that Z is also Y reflected over the line of symmetry of isosceles $\triangle BDC$.

Now, we are ready to tie everything together, with

$$DZ = DY = AX = EZ$$

We are basically done. Now, all that remains is to recognize $\triangle DZE$ is isosceles, and

$$\angle BDZ = 80°, \angle ZDE = \frac{180° - 80°}{2} = 50° \implies \angle BDE = 80° - 50° = 30°$$

and our answer is $\angle BDE = 30°$.

2 Solutions to Section 2 Problems

Solution to Problem 2.1.

Note that $\triangle ADP \sim \triangle CDB$. Then,

$$\frac{DP}{AP} = \frac{DB}{CB}$$

Likewise, $\triangle CDA \sim \triangle BDP$. So,

$$\frac{DP}{BP} = \frac{DA}{AC}$$

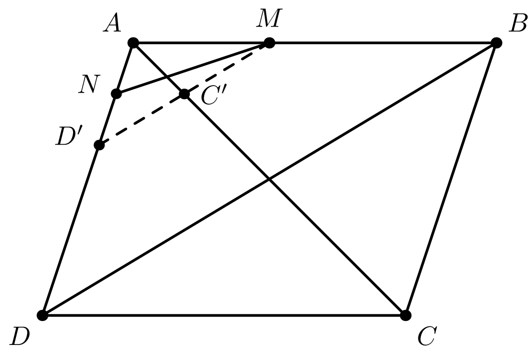
Adding the two results together (and using the fact that $AB = BC = CA$), we get

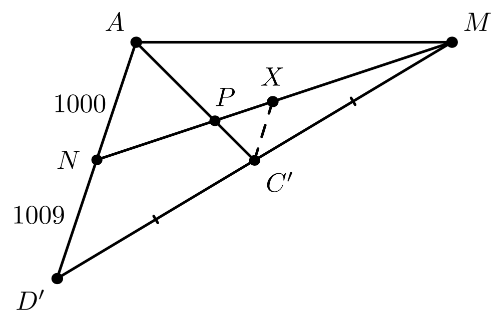
$$\frac{DP}{AP} + \frac{DP}{BP} = \frac{DA + DB}{AB} = 1 \implies \frac{1}{AP} + \frac{1}{BP} = \frac{1}{DP}$$

Solution to Problem 2.2.

The main annoying part of this problem is that the configuration is inside a parallelogram inside of a triangle. However, we can easily shift the problem so that our attention resides within a triangle.

We shrink $\triangle ABD$ down to the size of M. To be more precise, take D' so that $\frac{AD'}{AD} = \frac{AM}{AB} = \frac{17}{1000}$. Here, $\triangle AMD' \sim \triangle ABD$. Finally, let MD' intersect AC at C'.





We are now able to draw a diagram, and the scenario becomes familiar.

$$\frac{AN}{AD'} = \frac{1000}{2009} \implies \frac{AN}{ND'} = \frac{1000}{1009}.$$

Draw X on MN such that $NC' = AD'$.

$$\frac{AP}{PC'} = \frac{AN}{XC'} = \frac{2000}{1009} \implies \frac{AP}{AC'} = \frac{2000}{3009}.$$

Finally, we stretch back to the original parallelogram. We stretch by a factor of $\frac{1000}{17} \cdot 2$, so

$$\frac{AP}{AC} = \frac{2000}{3009} \cdot \frac{17}{2000} = 117$$

Solution to Problem 2.3.

The most obvious thing to do is ratios. Since K is the midpoint of PM and AC, $MAPC$ is a parallelogram. Then, $CP = AM = 180$.

Now, we know $AK : KL = 1$, and $AL : LB = 450 : 300$ by the angle bisector theorem. Can we find $PL : CP$?

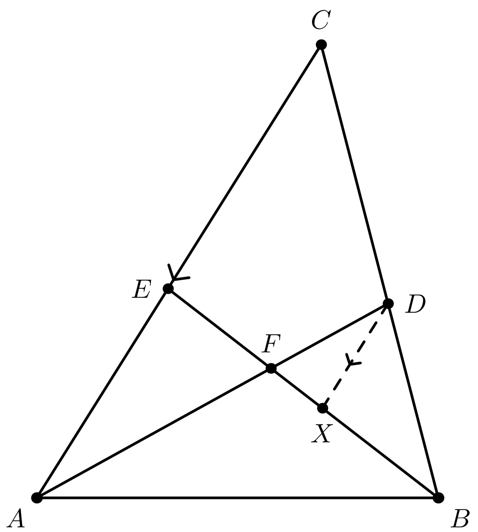
Indeed, this is routine. Draw X on KP such that $XL \parallel AC$. We have $\triangle BPL \sim \triangle BKA$, and $\triangle PXL \sim XKC$. Therefore,

$$\frac{XL}{KC} = \frac{XL}{AC} \implies \frac{PL}{CP} = \frac{BL}{AB} = \frac{300}{300+450}$$

Thus, $PL = \frac{300}{750} \cdot 180 = 72$.

A slightly more direct approach this problem is to consider $\triangle BPL \sim BMA$, which is basically the idea behind proving the angle bisector theorem.

Solution to Problem 2.4.



The parallel segment technique appears again. Draw X on BE so that $XD \parallel AC$. Due to the parallel condition, we have two similar triangles

$$\triangle BDX \sim \triangle BCE \quad \text{and} \quad \triangle AEF \sim DXF$$

So, we have

$$\frac{AF}{DF} = \frac{AE}{XD} \text{ and } \frac{XD}{EC} = \frac{BD}{BC} \implies \frac{AF}{FD} = \frac{AE}{EC} \cdot \frac{BC}{BD}$$

Now, it remains to find AE/EC and BC/BD. Here, we may use the angle bisector theorem.

$$\frac{AE}{EC} = \frac{AB}{BC} = \frac{6}{7}$$

$$\frac{BC}{BD} = \frac{BD+DC}{BD} = \frac{AB+AC}{AB} = \frac{6+8}{6}$$

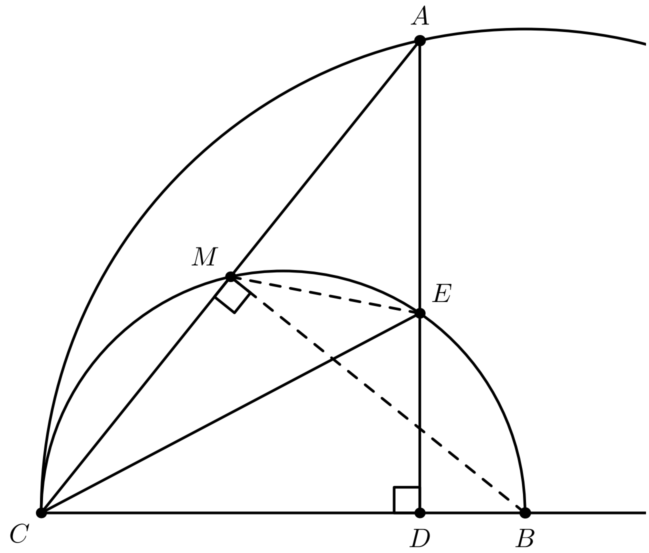
Finally, our answer is

$$\frac{AF}{DF} = \frac{6}{7} \cdot \frac{6+8}{6} = 2$$

Solution to Problem 2.5.

This problem is a more confusing problem. We can draw in circle O, but it doesn't seem like we can apply power of a point.

This time, we try to examine the structure of the problem. Let us fix BC. The locus of possible A is a circle around B with radius BC. After we chose A, D is the foot from A to BC. O is still the same circle, and E is the intersection of AD and O.



There must be a fixed ratio between AC and CE, for our problem to have a definitive answer. So, we intuitively must have similar triangles somewhere.

Let us intersect AC and O at M. It follows that $\angle CMB = 90^\circ$, since BC is a diameter. Then, M is the midpoint of base AC!

Now, there seems to be a pair of similar triangles formed. Indeed,

$$\angle CAE = 90^\circ - \angle ACD \quad \text{and} \quad \angle MEC = \angle MBC = 90^\circ - \angle ACD$$

So, by AA similarity

$$\triangle CME \sim \triangle CEA \implies \frac{CE}{CM} = \frac{CA}{CE} \implies CE = \sqrt{\frac{1}{2}CA^2} = \sqrt{2}$$

Solution to Problem 2.6.

Here's a problem that is quite scary looking, but certainly not hard. We just need to keep track of where everything is going.

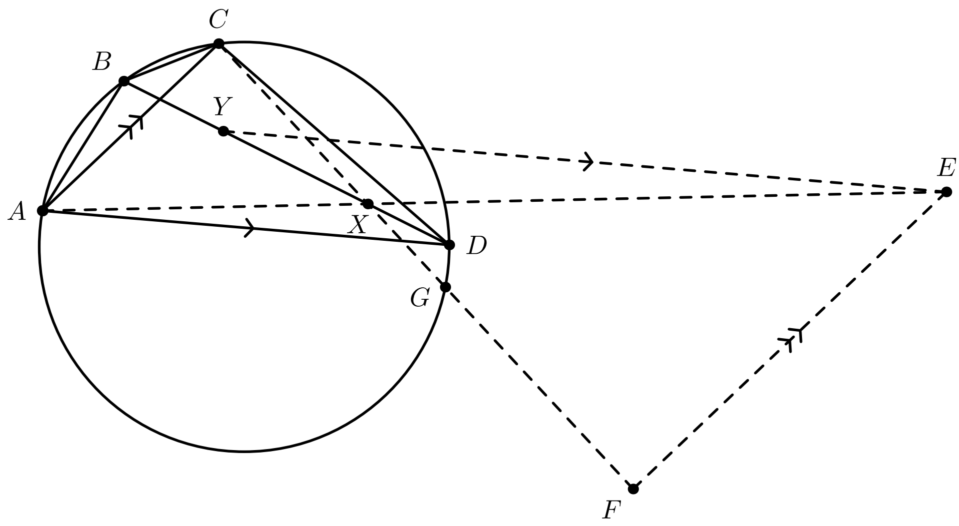
We want to find $XF \cdot XG$, and we do this by writing XF, XG in terms of given lengths separately. It is essential to draw a correct diagram here.

Since $AC \parallel EF$, we have

$$\triangle CXA \sim \triangle FXE \implies \frac{XF}{XE} = \frac{XC}{XA}.$$

Also, we have $EY \parallel AD$, which means

$$\triangle YXE \sim \triangle DXA \implies \frac{XE}{XA} = \frac{XY}{XD}.$$



Combining the two equations to cancel XE gives

$$XF = \frac{XC}{XA} \cdot \frac{XY}{XD} \cdot XA.$$

Now, using Power of a Point on X, we may find that

$$XC \cdot XG = XB \cdot XD \implies XG = \frac{XB \cdot XD}{XC}.$$

Conveniently, we find that

$$XF \cdot XG = XB \cdot XY = BD^2 \cdot \frac{XB}{BD} \cdot \frac{XY}{BD},$$

where all not nice terms cancel, and we are left with BD and some fractions we know.

Now, to find BD^2, we use the stronger form of Ptolemey's theorem (**Example 2.11**), and we obtain

$$BD^2 = \frac{(6+48)(16+18)}{(24+12)} = 51$$

To finish,

$$XF \cdot XG = 51 \cdot \frac{3}{4} \cdot \frac{16}{36} = 17.$$

Solution to Problem 2.7.

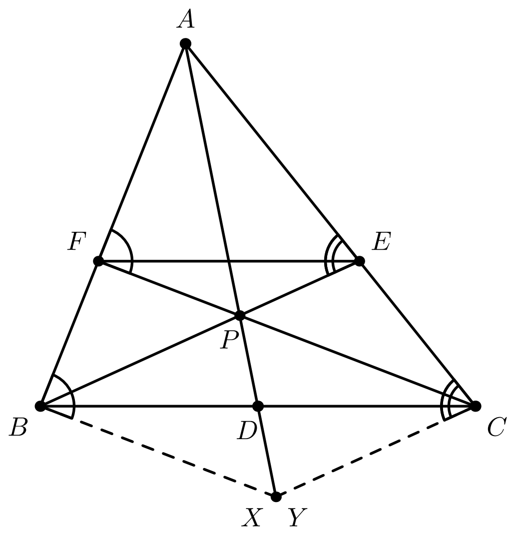
Now, the final problem for this section will be a USAMO problem. It isn't that hard, but quite rewarding, since it brings in many ideas of this handout (including some later in the handout). As so, it is worth revisiting this problem after you finish the handout.

From **Example 2.6**, we know that $MF = MC$ only if $AE \parallel CF$. However, in this problem we need to prove both if and only if directions. Is the if direction also true?

Indeed, it is, and we again need to use some ratios to force congruence. For convince, we will temporarily use the labeling of **Example 2.6**.

Let the line through B parallel to CP intersect AD at X. Likewise, let the line through C parallel to BP intersect AD at Y.

We need to show that $X = Y$ (which in turn will become P'). However, this is not obvious at all, and we don't have any direct techniques. (Remember, we can't use $BD = DC$, because that is what we want to prove!)



What we can do is to note that, by the parallel condition, we have

$$\triangle AFP \sim ABX \quad \text{and} \quad \triangle AEP \sim \triangle ACY.$$

Now,

$$\frac{AF}{FB} = \frac{AP}{AX} \quad \text{and} \quad \frac{AE}{AC} = \frac{AP}{AY}.$$

However, remember that $EF \parallel BC$ and $\triangle FAE \sim \triangle BAC$, so

$$\frac{AF}{FB} = \frac{AE}{AC} \implies \frac{AP}{AX} = \frac{AP}{AY}.$$

Now, we know $X = Y$, call it P', and $BPCP'$ is a parallelogram, since $BP' \parallel CP$ and $CP' \parallel BP$. Thus, $BD = DC$.

Returning back to the original problem, we now want $AE \parallel FC$ if and only if $MB \cdot MD = MC^2$.

The later condition, looks very much like similar triangles. Indeed,

$$\frac{MD}{MC} = \frac{MC}{MB} \iff \triangle MDC \sim \triangle MCB.$$

Now, a necessary and sufficient condition for $\triangle MDC \sim \triangle MCB$ is $\angle MCD = \angle MBC$ (which completes AA similarity).

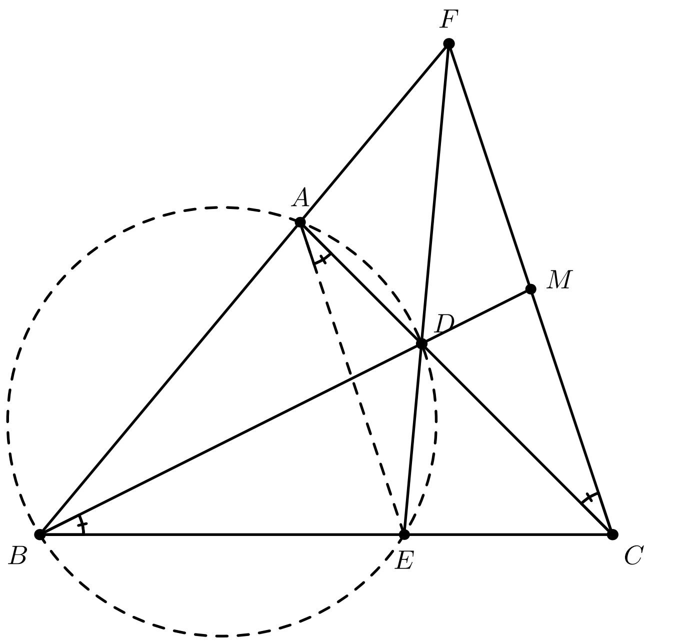
We also need to use the cyclic condition, and note that

$$\angle DBE = \frac{1}{2}\widehat{DE} = \angle DAE.$$

Finally, observe that $\angle DAE = \angle MBC = \angle MCD$ is just the condition for $AE \parallel FC$!

So, we can tie everything together:

$$MB \cdot MD = MC^2 \iff \angle MCD = \angle MBC \iff AE \parallel FC.$$



As an extra remark, you might have noticed that $MB \cdot MD = MC^2$ looks like Power of a Point. Indeed, you would be correct, and here MC is a "degenerate" line, where it only intersects the circle once. We will see more of this in **Example 5.2**.

3 Solutions to Section 3 Problems

Solution to Problem 3.1.

Construct B' on AB so that $DB'BC$ is a parallelogram. $B'B = DC = 39$, so $AB' = AB - BB' = 52 - 39 = 13$.

Since $DB' =$ CB $= 12$, and $DA = 5$, $\triangle ADB'$ is then a 5-12-13 right triangle. So, the altitude h from D to AB' must satisfy

$$\frac{AB' \cdot h}{2} = [ADB'] = \frac{5 \cdot 12}{2} \implies h = \frac{60}{13}$$

Therefore, we may find that

$$[ABCD] = [ADB'] + [DB'BC] = \frac{5 \cdot 12}{2} + 39 \cdot \frac{60}{13} = 210$$

Solution to Problem 3.2.

Without loss of generality, let $AB = 1$. Let $AE = x$. Then, $ED = EF = 1 - x$.

Since $\triangle BEF$ is equilateral, we can compute the side length of $\triangle BEF$ in two ways, with the Pythagorean Thereom on $\triangle EAB$ and $\triangle EDF$.

$$\sqrt{1^2 + x^2} = BE = EF\sqrt{(1-x)^2 + (1-x)^2} \implies x^2 - 4x + 1 = 0$$

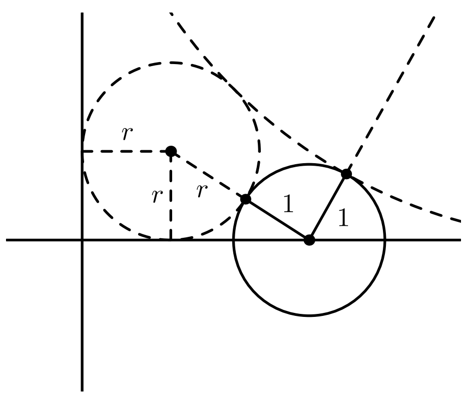
We can solve this, but it turns out that we don't need to, just yet. Let's first find the area ratio in terms of x.

$$\frac{[DEF]}{[ABE]} = \frac{DE \cdot DF}{AB \cdot AE} = \frac{1 - 2x + x^2}{x}.$$

Finally, we can finish with some nice algebraic manipulation.

$$x^2 - 4x + 1 = 0 \implies \frac{1 - 2x + x^2}{x} = \frac{[DEF]}{[ABE]} = 2.$$

Solution to Problem 3.3.



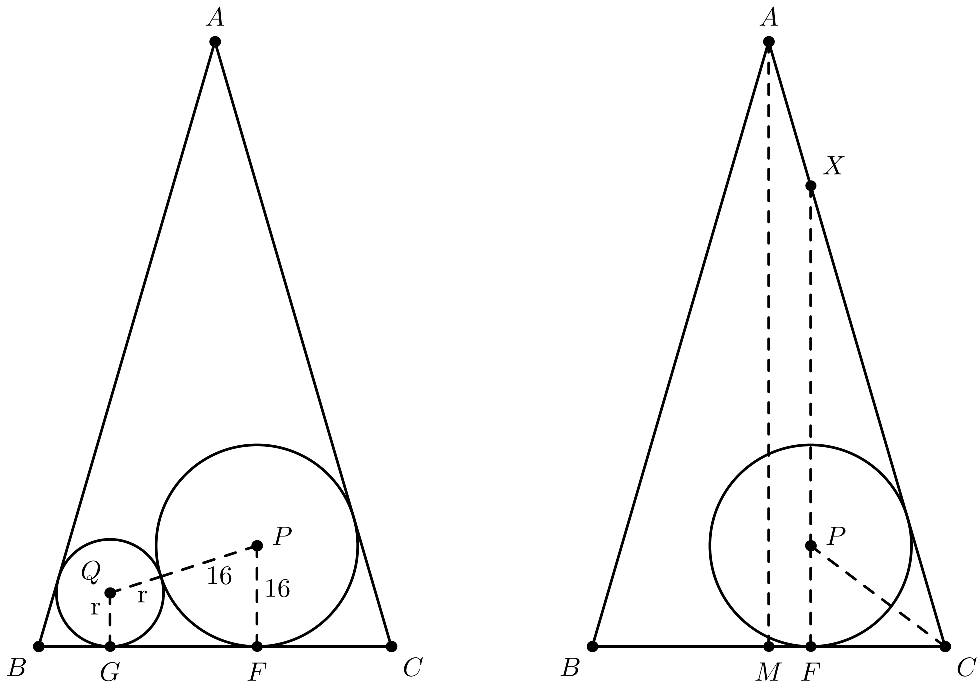
Once again, we try to set up a right triangle with the segment joining the centers of the two circles as the hypotenuse.

While the hypotenuse has length $1+r$, the legs parallel to the axes have lengths $r, \pm(3-r)$ depending on if the circle is larger or smaller. Regardless, we can setup

$$\begin{aligned}(3-r)^2 + r^2 &= (r+1)^2 \\ r^2 - 8r + 8 &= 0 \\ r &= \frac{8 \pm \sqrt{32}}{2}\end{aligned}$$

Therefore, we can quickly deduce that the sum of two possible $r_1 + r_2 = 8$.

Solution to Problem 3.4.



Let F, G be the point of tangency between circle P, Q and BC, respectively. By symmetry of tangents, observe that

$$\angle PCF = \angle QBG = \frac{\angle C}{2} = \frac{\angle B}{2}$$

If we want to express CF, BG in terms of r, we need to use the angle bisector trick again.

Let M be the midpoint of BC. $\triangle CMB$ is a 7-24-25 right triangle. Now, let PF intersect AC at X. $\triangle CMB \sim \triangle CFX$, and the later is also a 7-24-25 right triangle. By the angle bisector theorem,

$$\frac{PF}{PX} = \frac{CF}{CX} \implies \frac{PF}{XF} = \frac{CF}{CF+XC} \implies \frac{PF}{CF} = \frac{24}{7+25} = \frac{3}{4}.$$

Since P lies on the angle bisector of $\angle C$, we have $PF/FC = 3/4$. Similarly, $QG/GB = 3/4$. Thus, we can say that

$$GF = BC - FC - BG = 56 - \frac{4}{3} \cdot (16 + r).$$

Now, we have a right triangle with side lengths $(16-r), GF, (16+r)$. We use the Pythagorean, and solve for r:

$$\begin{aligned} GF^2 + (16-r)^2 &= (16+r)^2 \\ \left(56 - \frac{4}{3}\cdot 16 - \frac{4}{3}\cdot r\right)^2 &= 64r \\ 26^2 - 52r + r^2 &= 36r \\ r &= 44 - 6\sqrt{35} \end{aligned}$$

and we are done.

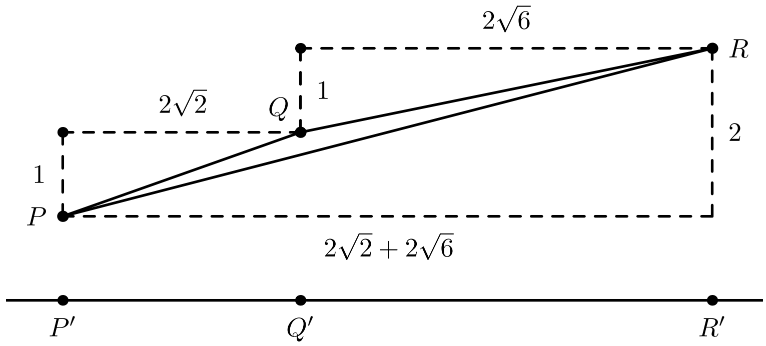
Solution to Problem 3.5.

Once again, we work in the "default" axes of ℓ and its perpendiculars.

It should be now routine to find $P'Q', Q'R'$. Indeed,

$$\begin{aligned} P'Q'^2 &= (2+1)^2 - (2-1)^2 \implies PQ = 2\sqrt{2} \\ Q'R' &= (3+2)^2 - (3-2)^2 \implies Q'R' = 2\sqrt{6} \end{aligned}$$

Now, the quickest way to solve is to carefully label all lengths in the direction of our axes.



Then, we simply subtract the area of the right triangles from the larger rectangle to get $[PQR]$.

$$[PQR] = 2\cdot(2\sqrt{2}+2\sqrt{6}) - 2\sqrt{2} - \sqrt{2} - \sqrt{6} - (2\sqrt{2}+2\sqrt{6}) = \sqrt{6}-\sqrt{2}.$$

Solution to Problem 3.6.

Let M, N be the midpoints of AB, CD, respectively. We find that

$$OM = \sqrt{25^2 - 15^2} = 20 \quad \text{and} \quad ON = \sqrt{25^2 - 7^2} = 24.$$

Now, P is a point such that $\angle OMP = \angle ONP = 90°$. Therefore, quadrilateral $OMPN$ must be cyclic, as opposite angles are supplementary. Moreover, since the angles are right, OP is a diameter.

We know the side lengths of $\triangle OMN$, so we can compute the circumradius of $\triangle OMN$.

$$R = \frac{20\cdot 12\cdot 24}{4\cdot\sqrt{28\cdot 4\cdot 8\cdot 16}} = \frac{45}{\sqrt{14}}$$

But OP is the diameter of the circumcircle of $\triangle OMN$, so $OP = 2R = 90/\sqrt{14}$.

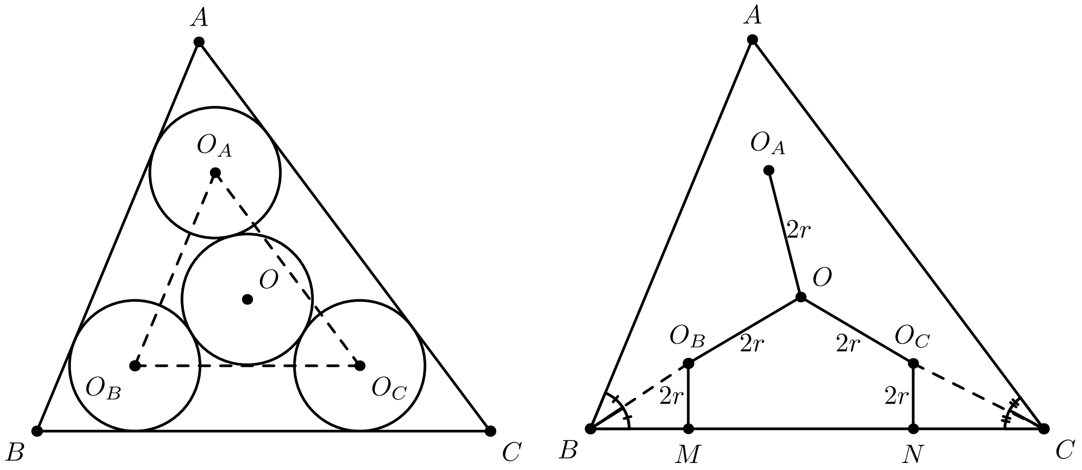
Solution to Problem 3.7.

Let the centers of $\omega_A, \omega_B, \omega_C, \omega$ be O_A, O_B, O_C, O, and let the radii of the four circles be r.

Note that the distance from O_B and O_C to BC is r, so $O_BO_C \parallel BC$. Likewise for O_AO_B and O_AO_C. Therefore,

$$\triangle O_AO_BO_C \sim \triangle ABC$$

Notice that $OO_A = OO_B = OO_C = 2r$, so O is actually the circumcircle of $\triangle O_AO_BO_C$.



Now, we want to find the proportionality factor k between $\triangle O_AO_BO_C$ and $\triangle ABC$, so we may say $2r/R = k$, where R is the circumradius of $\triangle ABC$.

Let the point of tangency between ω_B, ω_C and BC be M, N, respectively, We use the tangency trick one more time to deduce

$$\frac{O_BM}{BM} = \frac{12}{5+13} = \frac{2}{3} \quad \text{and} \quad \frac{O_CN}{CN} = \frac{12}{9+15} = \frac{1}{2}$$

So, we have the ratio of the side lengths in terms of r:

$$k = \frac{O_BO_C}{BC} = \frac{BC - BM - CN}{BC} = \frac{14 - \frac{7}{2}r}{14}$$

Finally, it is easy to find $R = 65/8$, which means that

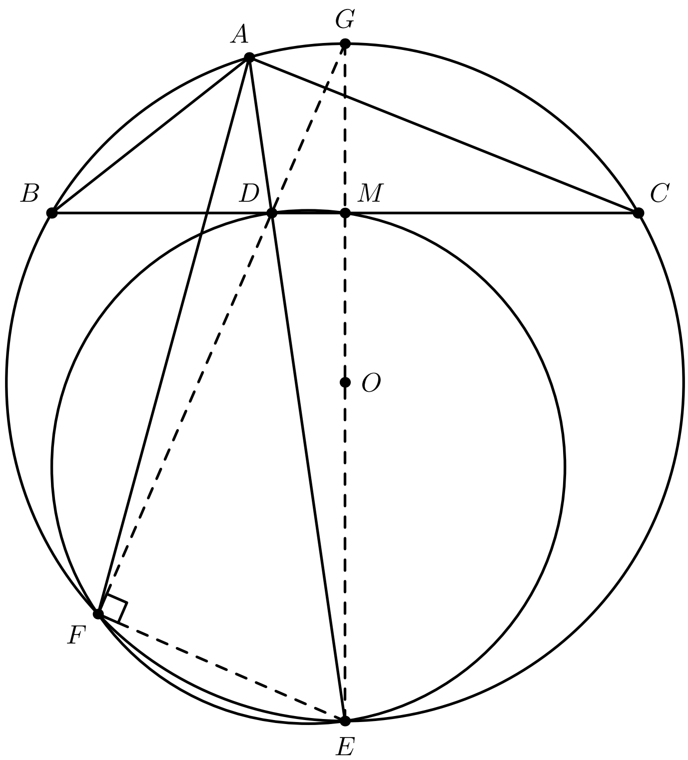
$$k = \frac{2r}{\frac{65}{8}} = \frac{14 - \frac{7}{2}r}{14} \implies r = \frac{260}{129}.$$

Solution to Problem 3.8.

This is another great problem illustrating an initial synthetic observation and bashing out the rest of the problem. In spirit of this section, of course we bash.

Since F lies on the circle γ with diameter DE, we need $\angle DFE = 90°$. Switching back to ω, DF must intersect ω again at a point diametrically opposite to E. Let this point be G, and let GE intersect CD at M.

Observe that $\triangle GDE \sim \triangle ADF$. Thus, to find AF we need GE, GD, AD. GE is just $2R$. AD is the length of the angle bisector, which can be obtained by Steward's.



Finally, GD can be computed by the Pythagorean Theorem on $\triangle GMD$. MD is simple, and we can use $GM = GO - MO = R - MO$, which we need another Pythagorean Theorem to find $MO^2 = R^2 - CM^2$.

And we are basically done! That was the setup; it remains to compute everything out.

First, we find the circumradius and DM.

$$R = \frac{3 \cdot 5 \cdot 7}{4 \cdot \sqrt{\frac{15}{2} \cdot \frac{9}{2} \cdot \frac{5}{2} \cdot \frac{1}{2}}} = \frac{7}{\sqrt{3}}.$$

$$CD = \frac{5}{8} \cdot 7 \ , \quad BD = \frac{3}{8} \cdot 7 \ \text{ and } \ DM = CD - CM = \frac{1}{8} \cdot 7$$

Now, we find AD with Stewart's.

$$\begin{aligned} AD^2 \cdot BC + DB \cdot DC \cdot BC &= AB^2 \cdot CD + AC^2 \cdot BD \\ AD^2 \cdot 7 + \frac{3 \cdot 5}{8^2} \cdot 7^3 &= 3^2 \cdot \frac{3}{8} \cdot 7 + 5^2 \cdot \frac{5}{8} \cdot 7 \\ AD^2 &= 15\left(1 - \frac{7^2}{8^2}\right) \\ AD &= \frac{15}{8} \end{aligned}$$

Lastly, we find GD.

$$
\begin{aligned}
GD^2 &= GM^2 + DM^2 = (GO - MO)^2 + DM^2 \\
&= \left(R - \sqrt{R^2 - CM^2}\right)^2 + DM^2 \\
&= \left(\frac{7}{\sqrt{3}} - \sqrt{\left(\frac{7}{\sqrt{3}}\right)^2 - \left(\frac{7}{2}\right)^2}\right)^2 + \left(\frac{7}{8}\right)^2 \\
&= \left(\frac{7}{\sqrt{3}} - \frac{7}{\sqrt{12}}\right)^2 + \left(\frac{7}{8}\right)^2 \\
&= 7^2 \cdot \left(\frac{1}{12} + \frac{1}{64}\right) \\
GD &= \frac{7}{8}\frac{\sqrt{19}}{\sqrt{3}}
\end{aligned}
$$

The algebra worked out quite nicely, so this computation really isn't that long. (especially if we just committed through)

Therefore, our final answer is

$$AF = GE \cdot \frac{GD}{AD} = 2 \cdot \frac{7}{\sqrt{3}} \cdot \frac{15}{8} \cdot \frac{8}{7}\frac{\sqrt{3}}{\sqrt{19}} = \frac{30}{\sqrt{19}}$$

4 Solutions to Section 4 Problems

Solution to Problem 4.1.

Since we are given that AD, EC are medians, we have enough information to find AP/PD and CP/PE. From Menelaus's , we have that

$$1 = \frac{AP}{PD} \cdot \frac{CD}{BC} \cdot \frac{BE}{AE} = \frac{AP}{PD} \cdot \frac{1}{2} \cdot \frac{1}{1} \implies \frac{AP}{PD} = 2$$

Be careful with the segments used in the equation, since the names given in this problem is different then our standard setup. (Alternatively, it may be more comfortable to just draw in the parallel line)

This is a useful fact to know, that the medians split each other in a two to one ratio. We now know that $AP = 4$, and likewise $CP = 3$.

Notice that $\triangle EPD$ is a right triangle. Then, we just need to compute the sum of area for four right triangles.

$$[ACDE] = [APC] + [CPD] + [DPE] + [EPA] = \frac{4 \cdot 3}{2} + \frac{3 \cdot 2}{2} + \frac{2 \cdot 1.5}{2} + \frac{1.5 \cdot 4}{2} = \frac{27}{2}.$$

Solution to Problem 4.2.

By Ceva's theorem, we have

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{AF} = 1$$

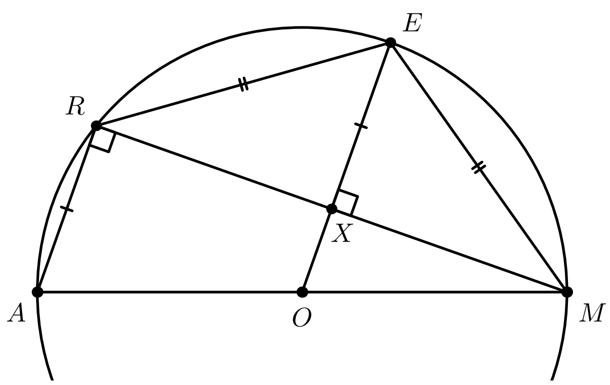
It is given that $CD = DB$ from the midpoint condition. Furthermore, by the angle bisector theorem

$$\frac{BF}{AF} = \frac{CB}{AC} = \frac{4}{5}.$$

Using Ceva's theorem gives us $AE/EC = 5/4$. Since $AC = 5$, it follows that $CE = 20/9$. Finally, we compute

$$EB = \sqrt{4^2 - \left(\frac{20}{9}\right)^2} = \frac{8\sqrt{14}}{9} \quad \text{and} \quad [ABC] = \frac{20\sqrt{14}}{9}$$

Solution to Problem 4.3.



First, we create altitudes. Let X be the foot of altitude from C to RM. Since AM is a diameter, $\angle RAM = 90°$, and RA is an altitude of RAM.

Compare $\triangle RAM$ and $\triangle REM$. Note that both triangles have share a base RM. Then, since their areas are equal, so are their altitudes, and $RA = EX$.

Since E lies on the angle bisector or $\angle RAM$, R is on the midpoint of arc RM, and $RX = XM$. So, E, X and the center of the circle O is collinear.

Now, $\triangle RAM \sim XOM$ by a scale factor of 2. So, $2XO = RA$. Denoting the radius of the circle to be r, we get

$$EO = r = EX + XO = \frac{3}{2}RA \implies RA = \frac{2}{3}r$$

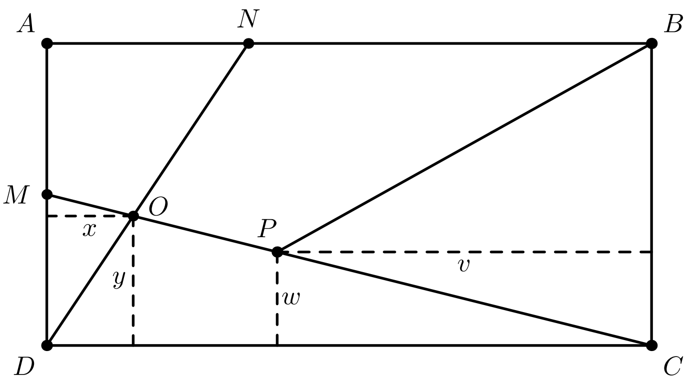
Finally, we need

$$RM = \sqrt{AM^2 - RA^2} = \sqrt{4r^2 - \left(\frac{2r}{3}\right)^2} = r\frac{4\sqrt{2}}{3}.$$

Since $r = 1$, the final area of

$$[MARE] = \frac{2}{3} \cdot \frac{4\sqrt{3}}{3} = \frac{8\sqrt{2}}{9}.$$

Solution to Problem 4.4.



Let $[ABCD] = 84 \cdot 42 = T$. Our strategy is to first express $[ADN], [CDM], [MDO]$ in terms of T to find $[BCON]$, since the former areas are easier to calculate (to find the area of a quadrilateral, what we really do is to find the areas of complementary triangles!).

It is easy to see that

$$[ADN] = \frac{1}{2} \cdot \frac{84}{3} \cdot 42 = \frac{T}{6} \text{ and } [CDM] = \frac{1}{2} \cdot 84 \cdot \frac{42}{2} = \frac{T}{4}.$$

$[MDO]$ is trickier, but nothing more then some proportions. Let the distance from O to A, CD be x, y, respectively. We have (without explicitly drawing in the altitudes and similar triangles)

$$\frac{x}{y} = \frac{AN}{AD} = \frac{84/3}{42} \text{ and } \frac{84 - x}{y} = \frac{DC}{DM} = \frac{84}{42/2}$$

This solves to $x = 12, y = 18$, which means that

$$[MDO] = \frac{1}{2} \cdot 18 \cdot \frac{42}{2} = \frac{T}{28}.$$

By complementary counting, we can see that (especially paying attention to how many times we count $[MDO]$)

$$[BCON] = T - \frac{T}{6} - \frac{T}{4} + \frac{T}{28} = \frac{13}{21}T.$$

Now, we need $\triangle BPC$ to have half that area. Again, letting the distance from P to BC, CD be v, w, we get

$$[BCP] = \frac{1}{2} \cdot v \cdot 42 = \frac{13T}{42} \implies v = 52.$$

To finish, we need to find w with

$$\frac{v}{w} = \frac{CD}{DM} = \frac{84}{42/2} \implies w = 13.$$

So, our final answer is

$$[CDP] = \frac{1}{2} CD \cdot w = 546.$$

Solution to Problem 4.5.

First, we note that we only need to consider $ABCD$, since the other side is the same by symmetry of parallelograms. It is given that $[ABD] = [ACD] = 84$. Since the two areas are equal, B, C must be equidistant from AD, so $ABCD$ forms a trapezoid.

The tricky part of this problem is to set up the numeric relations effectively. Let the intersection of AC, BD be G. Recall that in **Example 4.9** we implicitly showed that

$$[AGB] = [CGD] = \kappa[BGC] = \frac{1}{\kappa}[DGA],$$

where κ is the scale factor between the $\triangle BGC \sim \triangle DGA$ similarity.

Let us use this fact. Letting $[AGB] = A$, We need

$$A = \frac{A}{\kappa} + 10 \quad \text{and} \quad A + \kappa A = 84.$$

This solves to $(A, \kappa) = (12, 6)$ or $(35, 7/5)$. To minimize $[ABCD]$, we take $A = 12$, and the total area of $ABCDEF$ is

$$2 \cdot (12 + 12 + 2 + 72) = 196.$$

Solution to Problem 4.6.

It is not hard to find DM, MC using the angle bisector theorem. What we need is BF, FC. Then, we can use Menelauses's to finish.

It is unclear how to use the right angle. Instead, why don't we focus on the other half of the diagram - $\triangle ABD$. Indeed, $\triangle ABD$ isn't doing much, but we can make it so.

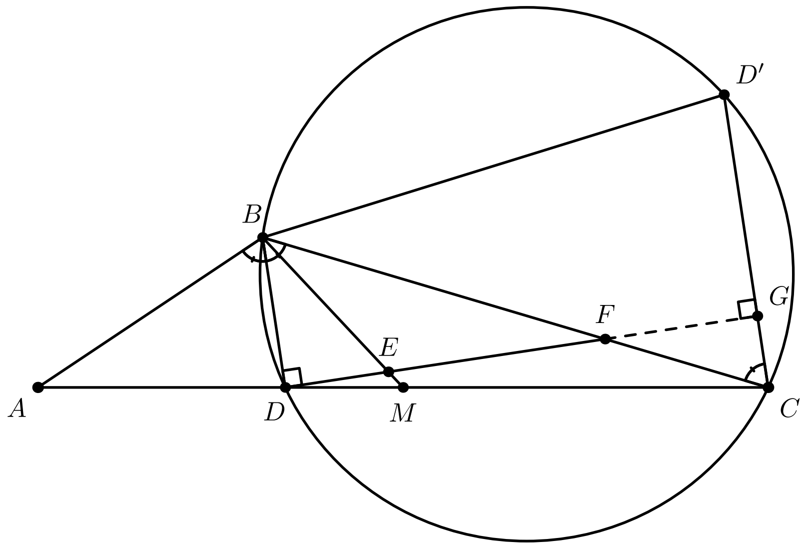
Recall **Example 2.12**? We are going to do the same thing, but with a twist. Take $\triangle ABD$, and map $B \to C, A \to B, D \to D'$.

Now, $\angle BD'C$ and $\angle BDC$ are supplementary, so we have cyclic $BDCD'$. Moreover, we also have $\angle D'CB = \angle DBA$. However, it is given, by the angle bisector condition, that $\angle DBA = \angle CBD$, so actually

$$\angle CBD = \angle D'CB \implies BD \parallel D'C!$$

Henceforth, we have created an isosceles trapezoid. Now, the right angle comes into play. In an isosceles trapezoid, the two altitudes create two congruent triangles and a rectangle. Let DG be altitude from D to $D'C$. We have

$$GC = \frac{1}{2}(D'C - BD)$$



We can find $D'C$. Since BA is mapped to CB, our scale factor is BC/BA.

$$GC = \frac{1}{2} \cdot BD \cdot \left(\frac{BC}{BA} - 1\right) = BD \cdot \frac{147}{720}.$$

Now, since $\triangle FGC \sim FDB$, we have

$$\frac{CF}{BF} = \frac{GC}{BD} = \frac{147}{720} \implies \frac{BC}{BF} = \frac{867}{720}.$$

We are ready to finish the problem.

$$DC = \frac{BC}{BC + BA} \cdot AC = \frac{507}{867} \cdot 780.$$

$$\frac{DM}{MC} = \left(\frac{507}{867} \cdot 780 - \frac{1}{2} \cdot 780\right) / \frac{1}{2} \cdot 780 = \frac{147}{867}.$$

At last, with Menelaus's

$$\frac{DE}{EF} = \frac{DM}{MC} \cdot \frac{CB}{BF} = \frac{867}{720} \cdot \frac{147}{867} = \frac{49}{240}.$$

Solution to Problem 4.7.

Let $AD \cap BE = X$, $BE \cap CF = Y$, $CF \cap AD = Z$. Consider $\triangle BXC$.

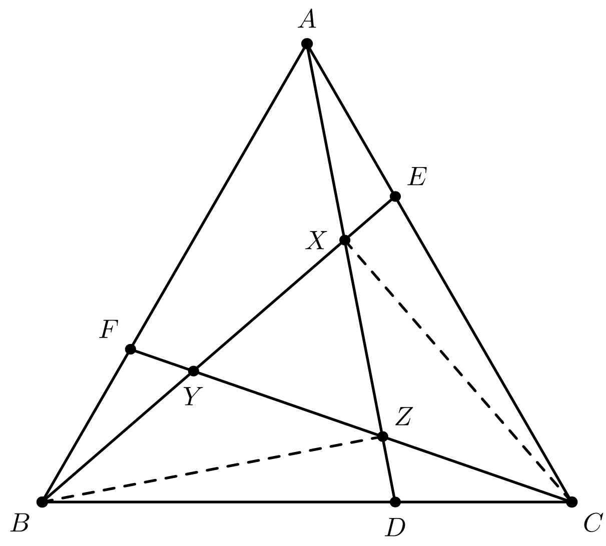
We first consider if $\triangle ABC$ is equilateral.

Since $BD = 2CD$, we have that $[BXD] = 2[CXD]$, and $[BZD] = 2[CZD]$, since the two pairs of triangles share the same height (third vertex).

This implies that $[BXZ] = 2[CXZ]$. However, by symmetry we have $\triangle BZY \cong \triangle CXZ$, which means that

$$[XYZ] + [BZY] = 2[CXZ] \implies [XYZ] = [CXZ].$$

The equal area condition implies that $YZ = ZC$, since the two triangles have the same altitude to YC.



We switch our attention to $\triangle BZY$ and $\triangle BZC$, that is on the other side of YC. Similairly, these two triangles have the same altitude and congruent bases ($YZ = ZC$), so

$$[BZC] = [BZY] = [XYZ] \implies [BYC] = 2[XYZ]$$

However, note that $\triangle ABC$ can be partitioned, by symmetry, into three triangles congruent to $\triangle BYC$ and the middle $\triangle XYZ$. So, it follows that the middle $\triangle XYZ$ has one-seventh the total area.

If $\triangle ABC$ is not equilateral, then our symmetry argument fails. Regardless, we can use Menelaus's, to find that

$$\frac{CZ}{ZF} = \frac{3}{4} \text{ and } \frac{DZ}{ZA} = \frac{1}{6}.$$

Our ratio argument works for all three cevians, and we then know that the three cevians are partitioned into a $3:3:1$ ratio, hence

$$YZ = ZC \text{ and } YX = YB.$$

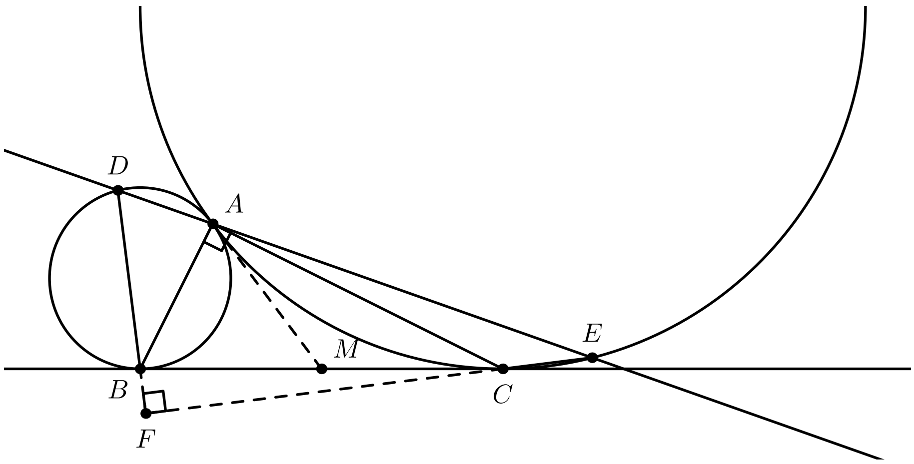
Note that $[XYZ] = [CXZ]$, since bases are equal and they share the same altitude. Furthermore, $[CXZ] = [CBZ]$, since they share the same base, and their altitudes (distance from X, B to YC) are equal (since $YX = YB$).

We can again partition $\triangle ABC$ into seven triangles of equal area, and once again $\triangle XYZ$ has one-seventh the total area.

5 Solutions to Section 5 Problems

Solution to Problem 5.1.

What we should first do again is to draw in the tangent line to $\mathcal{P}, \mathcal{Q}$ at A. Let the tangent intersect BC at M. We know that equal tangents imply $MB = MA = MC$, so $\triangle BAC$ is a right triangle.



Now, we also should use inscribed angles on the tangent angles. We have

$$\angle CEA = \angle MAC \text{ and } \angle BDA = \angle MAB.$$

This means that $\angle CEA + \angle BDA = \angle BAC = 90^\circ$ so if we let DB, EC intersect at F, then $\angle DFE = 90^\circ$. Furthermore,

$$\triangle DFE \sim \triangle BAC.$$

This is of course nice, but we need a better means of extrapolating the information.

First, we can say that $BACF$ is cyclic, with BC diameter, since both $\angle BAC$ and $\angle BFC$ are right angles. Next, we can angle chase

$$\angle CEA = \angle MAC = \angle MCA = \angle BFA,$$

where the second equality comes from isosceles $\triangle AMC$, and the last equality comes from cyclic $BACF$.

We also have $\angle ABF = 180^\circ = \angle ACF = \angle ACE$. Thus,

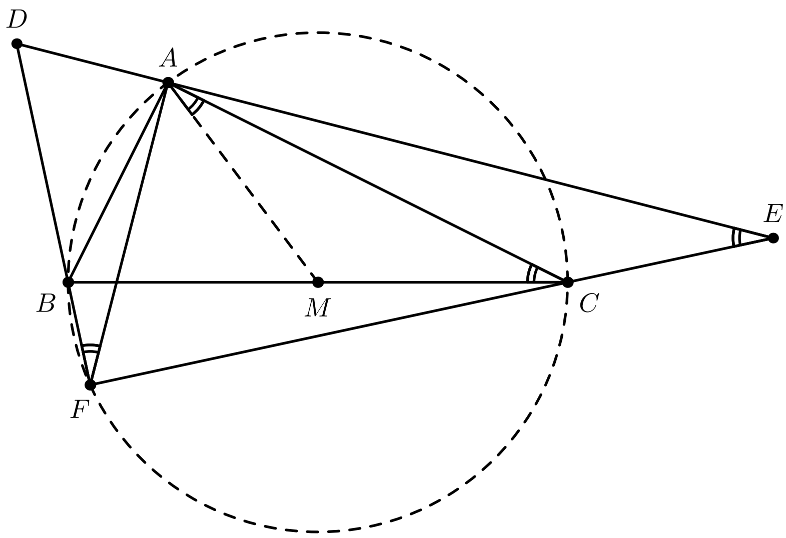
$$\triangle ABF \sim \triangle ACE!$$

Likewise, we have $\triangle ACF \sim \triangle ABD$, on the other side. Also, it happens that $AF \perp DE$, which can make drawing the diagram easier.

This is as far as we get without the area condition $[ABD] = [ACE]$. But first, it helps to to know that the similarity ratio for the two pairs of triangles above is AB/AC and AC/AB, so it helps knowing this value.

By now, we can easily find $BC = \sqrt{(4+1)^2 - (4-1)^2} = 4$. Hence $MB = MA = MC = 2$. The most direct (though overkill) method to find AB and BC, respectively, is probably the strong form of Ptolemey's.

$$AB^2 = \frac{(2 \cdot 1 + 2 \cdot 1)^2}{(2 \cdot 2 + 1 \cdot 1)} \text{ and } AC^2 = \frac{(2 \cdot 4 + 2 \cdot 4)^2}{(2 \cdot 2 + 4 \cdot 4)}$$



Upon solving, we obtain

$$AB = \frac{4}{\sqrt{5}}, AC = \frac{8}{\sqrt{5}}, \quad \text{and} \quad \kappa = \frac{AC}{AB} = 2.$$

Returning to the similar triangles, we note that the scale factor between $\triangle ABF$ and $\triangle ACE$ is κ. Hence,

$$[ABF] = \frac{1}{\kappa^2}[ACE]$$

However, it is given that $[ACE] = [ABE]$, so we also have

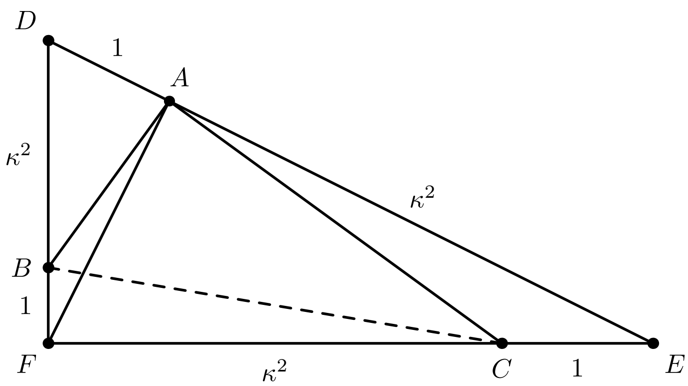
$$[ABF] = \frac{1}{\kappa^2}[ABD]$$

But now, these triangles share an altitude! So, we can actually deduce that

$$BF \cdot \kappa^2 = DB$$

We do the same with the other side, and obtain (reciprocally)

$$CF = CE \cdot \kappa^2$$



Finally, we note that

$$\frac{AE}{AF} = \frac{AF}{AD} = \frac{FE}{FD} = \kappa \implies DA \cdot \kappa^2 = AE.$$

Now, setting $\kappa = 2$, and $[DFE] = T$, we find

$$[ABD] = [BFC] = [ACE] = \frac{4}{25}T \implies [BAE] = \frac{13}{25}T.$$

But we have found AB, AC previously. So,

$$[ABD] = [ACE] = \frac{4}{13} \cdot [BAE] = \frac{4}{13} \cdot \frac{1}{2} \cdot \frac{4}{\sqrt{5}} \cdot \frac{8}{\sqrt{5}} = \frac{64}{65}.$$

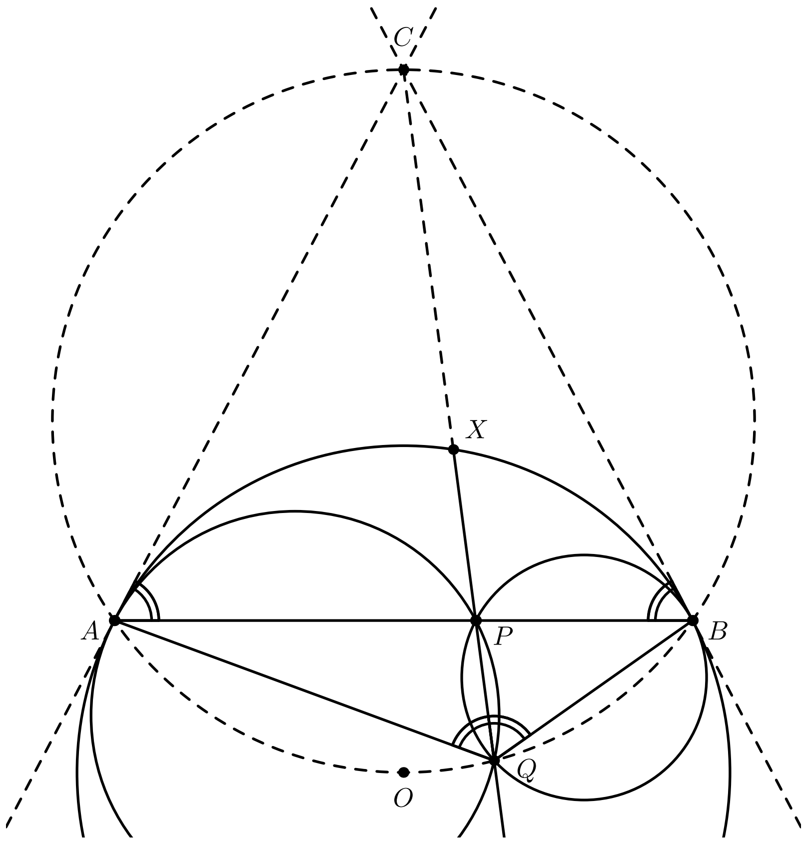
Solution to Problem 5.2.

The first matter on our hands is how to deal with the tangents condition. What we can say is that two circles tangent to each other are tangent to the same line at the point of contact.

Let CA, CB be the tangents to ω. Then, they are also tangents of ω_1, ω_2, respectively. By inscribed angles (**Example 5.2**), we have

$$\angle PAQ = \angle CAP = \frac{1}{2}\widehat{AB}_\omega \text{ and } \angle PQB = \angle CBP = \frac{1}{2}\widehat{AB}_\omega$$

Let $\frac{1}{2}\widehat{AB}_\omega = \theta$. Now, $\angle ACB = 180° - 2\theta$. However, $\angle AQB = 2\theta$, so $CAQB$ is cyclic.



Furthermore, we can note that $\angle AOB = \widehat{AB}_\omega = 2\theta$, so O is also on this circle, and CO must be a diameter.

Thus, we can note that $\angle OQP = 90°$, which actually says that Q is the midpoint of chord XY!

This is enough for us to finish the problem with power of a point, but we will also note a different approach.

This time, we draw in the centers O_1, O_2 of ω_1, ω_2. We know that A, O_1, O are colinear, due to our tangents properties. Likewise, B, O_2, O are colinear.

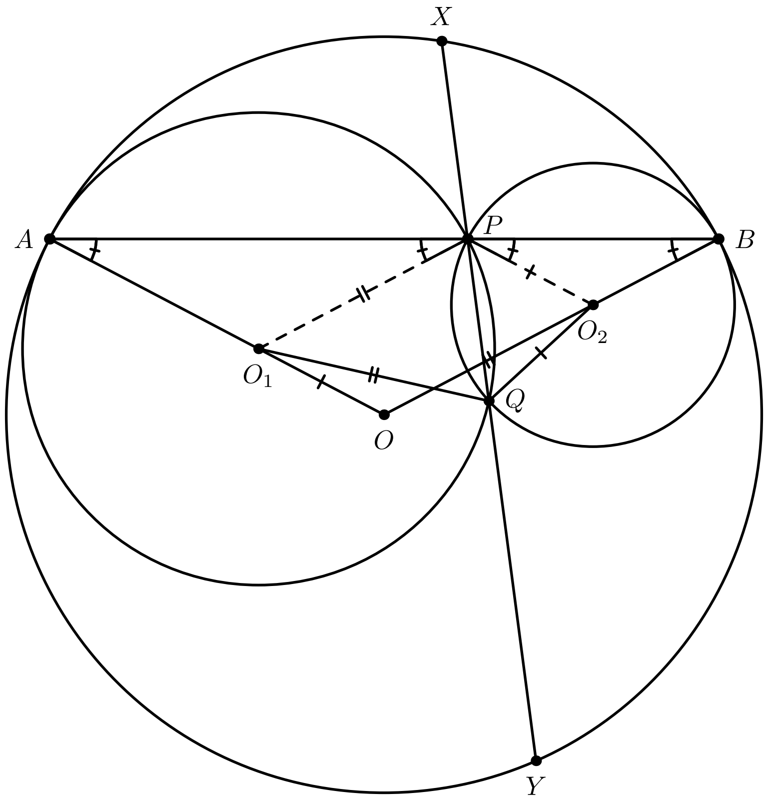
Next, we note that

$$\widehat{AP}_{\omega_1} = \widehat{AB}_{\omega} = \widehat{BP}_{\omega_2} = 2\theta.$$

So, we have that

$$\angle O_1AP = \angle O_1PA = \angle O_2PB = \angle O_2BP = 90^\circ - \theta.$$

This is enough to imply that O_1PO_2O is a parallelogram. So, $r_1 = O_1P = OO_2$ and $r_2 = O_2P = OO_1$. However, as Q is the other intersection of ω_1 and ω_2, we also have $O_1Q = r_1$ and $O_2Q = r_2$.



Doesn't this look familiar? Indeed, we have encountered a similar scenario in **Example 5.11**. What is important to notice is that O_1O_2OQ must be an isosceles trapezoid, which forces $OQP = 90^\circ$.

This time, we finish the problem. By Power of a Point on P,

$$AP \cdot PB = XP \cdot PY \implies 5 \cdot 3 = \left(\frac{11}{2} - PQ\right) \cdot \left(\frac{11}{2} + PQ\right)$$

So, it turns out that $PQ = \sqrt{61}/2$.

Solution to Problem 5.3.

Our first instincts is probably to use Power of a Point. We obtain

$$AP \cdot BP = 10 \cdot (25 + 15) \text{ and } AQ \cdot CQ = (10 + 25) \cdot 15.$$

However, there doesn't seem like much we can do after. Nevertheless, there is a very nice coincidence that allows us to continue.

Remember that we derived in **Example 5.7** that $\triangle ABC \sim \triangle AQP \sim \triangle DBP \sim \triangle DQC$? Indeed, let us extrapolate $\triangle AQP \sim \triangle DBP$:

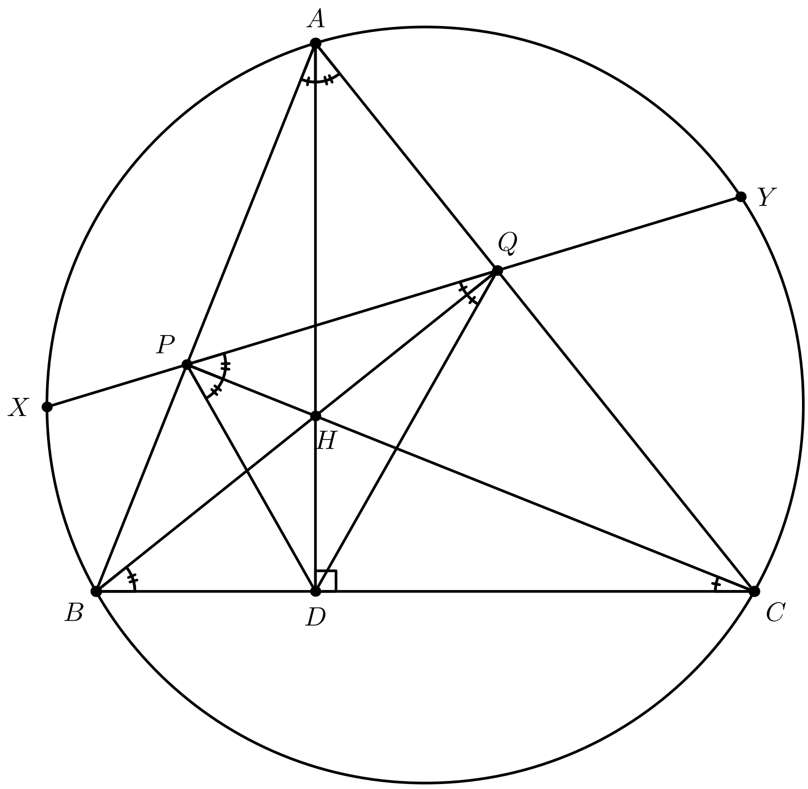
$$\frac{AP}{QP} = \frac{DP}{BP} \implies DP \cdot PQ = AB \cdot BP.$$

Very nice! Though we don't know AP, BP seperately, their product turns out to be useful enough.

We can do this to obtain $PD = 15, QD = 21$. Now, one viable strategy may be to look for PD/CA and QD/BA, so we can obtain $AB \cdot AC$. We have

$$\frac{PD}{CA} = \frac{BD}{BA} \text{ and } \frac{QD}{BA} = \frac{CD}{CA}.$$

Now, these ratios are nicely contained in right triangles. So, if we find another angle corresponding to $\angle BAD$ and $\angle DAC$, then we can examine the ratios in those similar triangles. (Okay, I know that this point we are basically doing trigonometry, please bear with me)

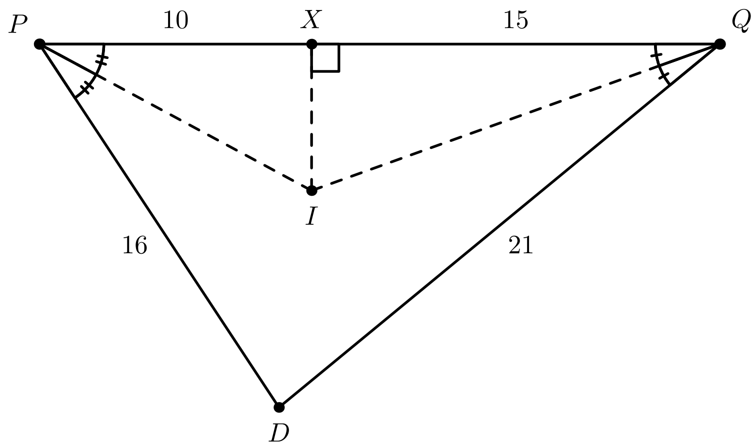


Luckily, these angles also appear in $\triangle PQD$. Let H be the orthocenter, and D be the foot of altitude from A to BC, we know that

$$90^\circ - \angle B = \angle BAD = \angle BCP = \angle DQH = \angle PQH$$

$$90^\circ - \angle C = \angle DAC = \angle CBQ = \angle DPH = \angle QPH$$

So, it turns out that the angle we need is half the angles of $\triangle PDQ$. So, if we look at $\triangle PDQ$, and create some right triangles from I, the incenter, then we should be good.



Currently, we have $\triangle PDQ$, with side lengths $16, 21, 25$. Then, the semiperimeter is $s = 31$. We can compute

$$A = \sqrt{31 \cdot 6 \cdot 10 \cdot 15} = 30\sqrt{31} \implies r = A/s = \frac{30}{\sqrt{31}}.$$

Now, let the perpendicular from I to PQ be X. We have

$$\triangle IQX \sim \triangle BAD \quad \text{and} \quad \triangle IPX \sim \triangle CAD.$$

In other words,

$$\frac{BD}{BA} = \frac{QI}{IX} \quad \text{and} \quad \frac{CD}{CA} = \frac{PI}{IX}$$

Tying everything together, we have

$$\begin{aligned} AB \cdot AC &= PD \cdot QD \cdot \frac{QI}{IX} \cdot \frac{PI}{IX} \\ &= \frac{16 \cdot 21}{900/31} \cdot \sqrt{15^2 + \frac{900}{31}} \cdot \sqrt{10^2 + \frac{900}{31}} \\ &= \frac{16 \cdot 21}{900/31} \cdot \frac{10 \cdot 15}{31} \cdot \sqrt{31+9} \cdot \sqrt{31+4} \\ &= 560\sqrt{14} \end{aligned}$$

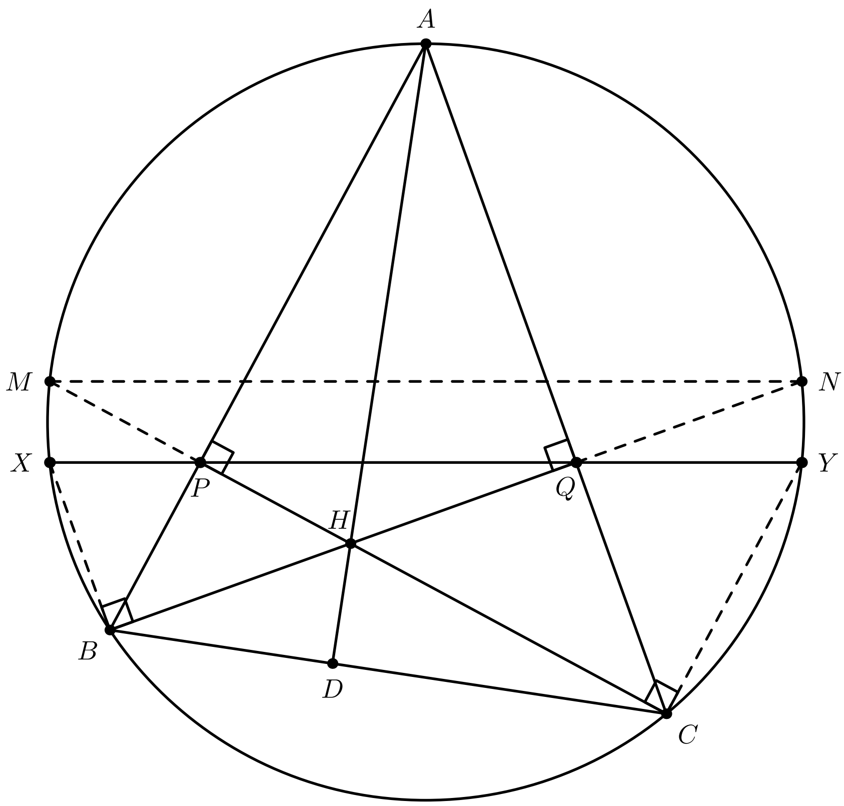
There is also a cleaner way to do this problem, if we find something special about the given values. We notice that $XP + QY = PQ$. Could that mean something nice?

What we do is reflect H, the orthocenter, over P, Q to M, N, respectively. We know that M, N lie on the circumcenter, from **Example 5.11**. Now, $\triangle MHN \sim \triangle PHQ$, with scale factor 2.

So, we now have $MN = 2 \cdot 25 = 50$. But $XY = 50$ as well, and the two lines are parallel. This forces the two chords to be reflections of each other over the center of the circle! Then, we have that XN, YM are diamaters of the circle.

Now, XB, AQ are both perpendicular to BN, so $XB \parallel AQ$. Likewise, $YC \parallel AP$. This allows us to form similar triangles. Combined with Power of a Point, we have enough equations to solve AB, AC directly.

$$AP \cdot BP = 10 \cdot (25 + 15) \quad \text{and} \quad \frac{AP}{BP} = \frac{25}{10}.$$



$$AQ \cdot CQ = (10 + 25) \cdot 15 \text{ and } \frac{AQ}{CQ} = \frac{25}{15}.$$

Our final answer is, again,

$$AB \cdot AC = 7\sqrt{40} \cdot 8\sqrt{35} = 560\sqrt{14}.$$

Solution to Problem 5.4.

After drawing in XY, it seems like XY might be antiparallel to BC, as we are quite used to seeing. Indeed, this is the case, and let us prove this. Have XY intersect AB, AC at P, Q, respectively.

By the tangency condition, we have $\angle QHC = \angle HBC$. Furthurmore, we also have $\angle ABH = \angle HCQ = 90^\circ - \angle A$. Adding the two, we have

$$\angle B = \angle ABE + \angle EBC = \angle QHC + \angle HCQ = 180^\circ - \angle HQC = \angle AQH,$$

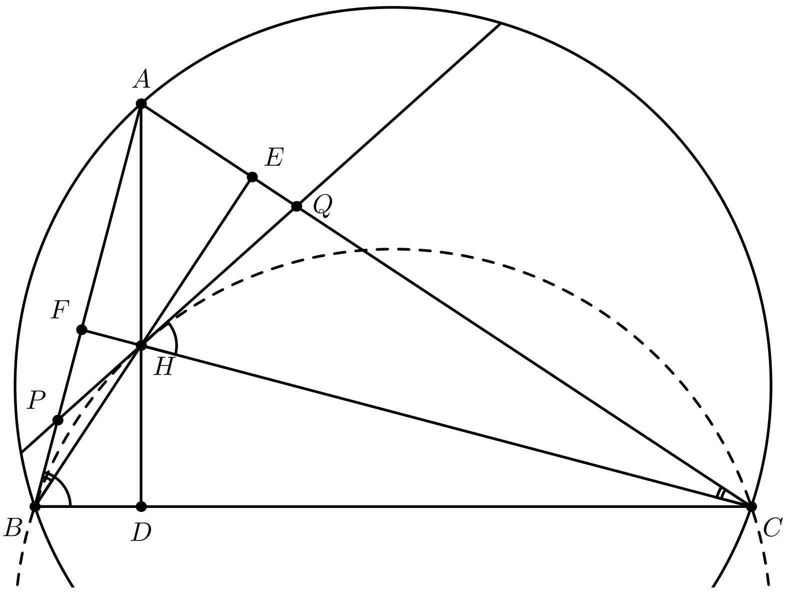
as desired. Thus, we have $\triangle APQ \sim \triangle ACB$.

Unfortunately, no more nice angle condition exists for X, Y. (Attempts to find some cyclic quadrilaterals with X, Y will just not work). Nevertheless, we can still invoke some length relations to use the information $HX = 2, HY = 6$.

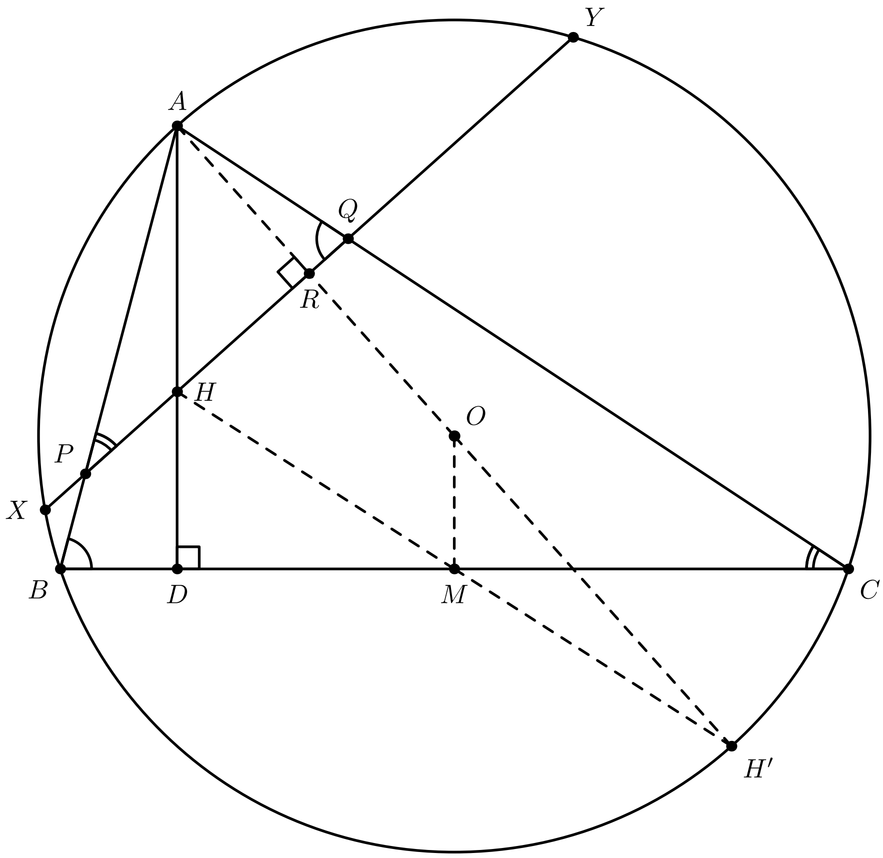
Power of a Point might be first to come to mind. If we extend AD to meet $\odot ABC$ at G, then $2 \cdot 6 = AH \cdot HG$, or $HG = 4$. Remembering **Example 5.11** and reflecting the orthocenter, we can instead say $HD = 2$.

Let R be the foot of altitude from A to PQ. We claim that A, R, O are colinear. First, we need to note that

$$\angle BAH = 90^\circ - \angle B = \angle CAO.$$



Now, since $\angle ADB = \angle ARQ = 90^\circ$ and $\angle ABD = \angle AQR$, it follows that $\angle BAD = \angle QAR$. Hence, $\angle CAO = \angle CAR$.



This is now useful because R must be the midpoint XY. We know that $HR = 2$, and $XR = RY = 4$. Now, $AR = \sqrt{AH^2 - HR^2} = 5$. We can set up an equation for OR to find R, the circumradius of $\triangle ABC$.

$$OA - AR = OR = \sqrt{OX^2 - OR^2} \implies R - \sqrt{5} = \sqrt{R^2 - 4^2} \implies R = \frac{21}{2\sqrt{5}}.$$

We know AD, so to find the area of $\triangle ABC$, we need the base length BC. Fortunately, we can find OM, the distance from O to BC, where M is the midpoint of BC.

Recall, again from **Example 5.11**, that we showed the reflection of H over M, H' is on $\odot ABC$. We also noted that AH' is a diameter. Now, we have that $\triangle H'MO \sim \triangle H'HA$, with a scale factor of 2. So, $OM = AH/2$.

That's enough.

$$BC = 2\sqrt{R^2 - OM^2} = 2\sqrt{\left(\frac{21}{2\sqrt{5}}\right)^2 - \left(\frac{3}{2}\right)^2} = 3 \cdot \sqrt{\frac{44}{5}}.$$

Our final answer is

$$[ABC] = \frac{1}{2} \cdot AD \cdot BC = 3\sqrt{55}.$$